Cognitive bases of spontaneous shortcut use in primary school arithmetic

DISSERTATION

Zur Erlangung des akademischen Grades doctor rerum naturalium

(Dr. rer. nat.) im Fach Psychologie

Eingereicht an der Lebenswissenschaftlichen Fakultät

der Humboldt Universität zu Berlin

von Dipl. Psych. Claudia Godau

Präsident der Humboldt-Universität zu Berlin: Prof. Dr. Jan-Hendrik Olbertz

Dekan der Lebenswissenschaftlichen Fakultät: Prof. Dr. Richard Lucius

Gutachter_innen:

1. Jun. Prof. Dr. Robert Gaschler, Universität Koblenz-Landau
2. Prof. Dr. Hilde Haider, Universität Köln
3. Dr. André Knops, Humboldt-Universität zu Berlin

Tag der Verteidigung: 12.01.2015

Abstract

Flexible use of task-appropriate solving strategies is an important goal in mathematical education and educational standard of elementary school mathematics. Children need to decide spontaneously whether they calculate arithmetic problems the usual way or whether they invest time and effort to search for shortcut options and apply them. The focus of the current work lies on how students can be supported in spotting and applying shortcut strategies flexibly. Therefore, I investigated contextual factors that influence the spontaneous usage of shortcuts, as well as the transfer between them. Cognitive theories about how mathematical concepts and strategies develop were combined with findings from research on expertise, which disclose differences between the flexibility of experts and novices. Mathematical concepts develop in an iterative fashion and the relationship between conceptual and procedural knowledge is bidirectional (Rittle-Johnson, Siegler, & Alibali, 2001). Improvements in understanding the concept can lead to improvements in strategy use (procedural knowledge) and vice versa. During mathematical development and with increasing expertise these two forms of knowledge increasingly integrate with each other (Haider et al., 2014). Successfully spotting and applying a shortcut might thus benefit from factors activating conceptual and/or procedural knowledge. I present contextual factors (*instruction, association and estimation*), which support or hinder spontaneous strategy use, using shortcuts based on commutativity ($a + b = b + a$) as a test case. In Journal Article 1, my colleagues and I have hypothesized and empirically investigated that instruction can hinder the flexible change of different shortcuts. Results showed descriptively that a group of children applying a shortcut spontaneously showed a better transfer of knowledge compared to a group that received an instruction. I discuss the advantages and disadvantages of spontaneous and instructed strategy use. In Journal Article 2, we investigated and demonstrated that children search for shortcuts spontaneously. We hypothesized and found that the use of two different shortcuts is associated, if the two shortcuts are based on the same principle. The link via the concept helped younger children to use diverse shortcuts spontaneously. In Journal Article 3, we tested why commutativity-based shortcuts in arithmetic might be used more frequently if children have worked on an estimation task before. A wealth of research shows that estimation is supportive for flexibility and transfer. This research did not consider superficial forms of transfer. For instance, long-range eye movements induced by an estimation task rather than estimation per se might provoke flexibility in arithmetic problems. My colleagues and I tested this account and found that changed fixation patterns did not lead to higher shortcut use. In Journal Article 4, we explored whether commutativity is used spontaneously in an estimation task. The results indicate that adults used commutativity in the processing of briefly presented bar graphs spontaneously. Overall, the dissertation shows that spontaneous strategy use can be supported by some contextual factors and impeded by others. These contextual factors can, in principle, be controlled in school environment.

Keywords: numerical cognition, spontaneous strategy application, commutativity

Zusammenfassung

Aufgabengeeignete Rechenstrategien flexibel zu nutzen ist ein wichtiges Ziel mathematischer Bildung und Bestandteil der Bildungsstandards der Grundschulmathematik. Kinder sollen spontan entscheiden, ob sie arithmetische Aufgaben in üblicher Weise berechnen oder ob sie Zeit und Aufwand investieren, um nach Vereinfachungsstrategien zu suchen und diese anzuwenden. Der Schwerpunkt der aktuellen Arbeit ist, wie Schüler beim flexiblen Erkennen und Anwenden von Vereinfachungsstrategien unterstützt werden können. Ich untersuchte daher Kontextfaktoren, welche die spontane Nutzung von Vereinfachungsstrategien und den Transfer zwischen ihnen beeinflussen. Kognitive Theorien über die Entwicklung von mathematischen Konzepten und Strategien wurden mit Erkenntnissen aus der Expertise Forschung verbunden, welche die Unterschiede in der Flexibilität zwischen Experten und Novizen offen legen. Mathematische Konzepte entwickeln sich iterativ und die Beziehung zwischen konzeptuellem und prozeduralem Wissen ist bidirektional (Rittle-Johnson et al., 2001). Verbesserungen im Verständnis des Konzeptes führen zu Verbesserungen in der Strategienutzung (prozedurales Wissen) und umgekehrt. Während der mathematischen Entwicklung und mit mehr Erfahrung integrieren diese beiden Formen des Wissens zunehmend (Haider et al., 2014). Erfolgreiches Erkennen und Anwenden einer Vereinfachungsstrategie könnte somit von Faktoren, die konzeptionelles und/oder prozedurales Wissen aktivieren, profitieren. Am Beispiel von Vereinfachungsstrategien, die auf dem Kommutativgesetz ($a + b = b + a$) basieren, präsentiere ich drei Kontextfaktoren (*Instruktion, Assoziation und Schätzen*) die spontanen Strategiegebrauch unterstützen oder behindern. Im 1. Artikel haben meine Kollegen und ich die Hypothese aufgestellt und empirisch untersucht, ob Instruktionen den flexiblen Wechsel verschiedener Vereinfachungsstrategien behindert. Die Ergebnisse zeigten deskriptiv einen besseren Transfer des Wissens in der Gruppe von Kindern, welche die Vereinfachungsstrategien spontan anwenden konnten im Vergleich zu der Gruppe, die eine Instruktion erhalten hatte. Ich diskutiere die Vor- und Nachteile der spontanen und instruierten Anwendung von Rechenstrategien. Im 2. Artikel untersuchten und zeigten wir, dass Kinder spontan nach Vereinfachungsstrategien suchen. Wir stellten die Hypothese auf und fanden, dass zwei verschiedene Vereinfachungsstrategien assoziert sind, wenn sie beide auf dem gleichen mathematischen Prinzip beruhen. Jüngeren Kindern half diese Verbindung über das Konzept die verschiedenen Vereinfachungsstrategien spontan zu nutzen. Im 3. Artikel testeten wir warum arithmetische Vereinfachungsstrategien, die auf dem Kommutativgesetz basieren, mehr benutzt werden, wenn die Kinder vorher Schätz-Aufgaben bearbeitet haben. Eine Vielzahl von Untersuchungen im Forschungsgebiet zeigen, dass Schätzen Flexibilität und Transfer unterstützt. Diese Forschung hat nicht oberflächliche Formen des Transfers betrachtet. Zum Beispiel provozieren vielleicht Langstrecken-Augenbewegungen, die durch Schätz-Aufgaben induziert wurden, statt Schätzung an sich Flexibilität in späteren Rechenaufgaben. Meine Kollegen und ich testeten diese Annahme und stellten fest, dass veränderte Muster der Fixationen nicht zu einer höheren Verwendung von Vereinfachungsstrategien führen. Im 4. Artikel untersuchten wir ob das Kommutativgesetz spontan in einer Schätz-Aufgabe verwendet wird. Die Ergebnisse zeigten, dass Erwachsene das Kommutativgesetz in der Verarbeitung von kurz präsentierten Graphiken spontan nutzen. Insgesamt zeigt die Dissertation, dass spontane Strategienutzung durch bestimmte Kontextfaktoren unterstützt und durch Andere behindert werden kann. Diese Kontextfaktoren können im Prinzip in der Schulumgebung gesteuert werden.

Schlagwörter: numerischen Kognition, spontane Strategieanwendung, Kommutativgesetz

Acknowledgements

I would like to express my appreciation to all those who provided me the possibility to complete this dissertation. A special gratitude I give to my advisor Professor Dr. Robert Gaschler for his contribution in stimulating suggestions and encouragement. In addition, a thank you to Professor Hilde Haider and her research group for supplying an additional forum for discussing approaches and results. This work was supported by Grant FR 1471/12-1 from the Deutsche Forschungsgemeinschaft (DFG) as well as by the Berlin Cluster of Excellence Image Knowledge Gestaltung (www.interdisciplinary-laboratory.hu-berlin.de).

Publications Included

Godau, C., Haider, H., Vaterrodt, B., Frensch, P. A., & Gaschler, R. (submitted). The downside of direct instruction – Instructing one shortcut can hinder spontaneous usage of another arithmetic shortcut based on the same mathematical principle. *Frontline Learning Research, Mai 2014.*

Godau, C., Haider, H., Hansen, S., Schubert, T., & Gaschler, R. (2014). Spontaneously spotting and applying shortcuts in arithmetic—a primary school perspective on expertise. *Cognition*, *5*, 556. doi:10.3389/fpsyg.2014.00556

Godau, C.*, Wirth, M., Hansen, S., Haider, H., & Gaschler, R. (2014). From Marbles to Numbers—Estimation Influences Looking Patterns on Arithmetic Problems. *Psychology*, *05*(02), 127–133. doi:10.4236/psych.2014.52020

Godau, C., Haider, H., & Gaschler, R. (submitted). Commutativity at first glance - mathematical and perceptual principles in bar-graph processing. *Journal of Cognitive Psychology, April 2014*.

* The first two authors shared first authorship and contributed equally to this work.

Table of contents

Introduction

Improving mathematics and science education has been a matter of international concern. Psychology plays a vital role in this enterprise (Newcombe et al., 2009). Comparisons are often conducted in educational school psychology internationally. There are more general and thematically broad investigations like PISA (Programme for International Student Assessment), as well as investigations with relatively specific themes on certain competencies. The TIMSS (trends in international mathematics and science study) 2011 investigated fourth and eighth grade school children in mathematics and MINT[1] - disciplines. Cognitive domains were also considered in these investigations – for example *knowing*, *applying*, and *reasoning*. The results of the TIMSS showed that German fourth graders are better in *reasoning* than in *knowing* (Mullis, Martin, Foy, & Arora, 2012). Specifically, the average score that German fourth graders received for *knowing* – which refers to the student's knowledge base of mathematics facts, concepts, tools, and procedures – was significantly lower than their overall mathematic average score (Mullis et al., 2012). Their average score for *reasoning* – which refers to going beyond the solution of routine problems to encompass unfamiliar situations, complex contexts, and multi-step problems –was, however, significantly higher than the overall average mathematics score. The student's ability to apply knowledge and to make use of conceptual understanding in a problem situation is also measured in TIMSS. *Applying* was meant as applying basic knowledge in straightforward situations, which was average (Bos, Wendt, Köller, & Selter, 2012). German fourth graders are good at *reasoning*, but there seems to be more potential for development in the *application* of knowledge.

Consider, as an example, children working on a list of three-addends problems during a mathematics lesson. One bench neighbour might ask the other, why he/she is already done. The answer could be, because he/she used a shortcut and did not need to calculate every problem the standard way. But why are shortcuts sometimes spotted and applied, but neglected in other situations? And how could students be supported in spotting and applying shortcuts flexibly?

In this dissertation I investigate how elementary school children can be supported in applying their knowledge, especially applying appropriate strategies in addition problems in a

[1] MINT - fields of mathematics, computer science, natural sciences and technology (German version of STEM - fields of sciences, technology, engineering and mathematics)

spontaneous and flexible manner. The task-appropriate use of flexible solving strategies is formulated as the ultimate ambition in the educational standard for mathematics in elementary school (State Institute for School Development Stuttgart & Ministry of Culture, Youth and Sport of the State of Baden-Württemberg, 2004). The usage of shortcuts is mentioned explicitly as a requirement for grades 1 and 2. The application of *commutative, associative,* and *distributive* laws is dealt with in grades 3 and 4 (Ministry of Education, Youth and Sport of the State of Brandenburg, Senate Administration for Education, Youth and Sports Berlin, Senator for Education and Science Bremen, & Ministry of Education, Science and Culture Mecklenburg-Vorpommern, 2004). Different scientific approaches overlap in the research on mathematics education – didactic issues like how children should be taught, as well as issues from the domain of cognitive psychology like cognitive processes of learning and understanding. I will focus on enhancing our understanding of cognitive processes involved in the educational goals discussed above. In the dissertation I will investigate the shortcut use based on commutativity in primary school arithmetic, which is one of the educational goals to be achieved before entering secondary school.

Commutativity as well as associativity and distributivity are part of the basic rules of algebra. Commutativity justifies changing the order or sequence of the operands within an expression (e.g. $a + b = b + a$) for addition and multiplication but not for subtraction and division. Commutativity enables students to change the order of addends within a problem flexibly. By using a commutativity-based shortcut students should save time. For instance, when presented with the problem $8 + 5 + 7 = ?$, and afterwards with the problem $5 + 7 + 8 = ?$, they do not need to calculate the second problem. In the current work, commutativity is used as a test case for investigating flexibility and transfer of knowledge. Furthermore, this principle is very relevant, because understanding the concept of commutativity is linked to students' understanding of addition as a binary rather than as a unary operation (Baroody & Gannon, 1984). The binary view of the addition of two numbers would, for instance, interpret $2 + 4$ as summing up two independent cardinalities, 2 and 4. The unary view would interpret $2 + 4$ as the addition of 4 more units to 2. In the unary view, one is added to the other, rather than that they are added together. Therefore, the two addends play an asymmetric role.

How can elementary school children be supported in using appropriate strategies spontaneously and flexibly in addition problems? Recent studies have focused on the spontaneous recognition of mathematical aspects in different situations (Hannula & Lehtinen, 2005; Hannula, Lepola, & Lehtinen, 2010; McMullen, Hannula-Sormunen, & Lehtinen, 2013). Some children pay more attention to the number of objects or events in their everyday environment than others already during early child development (pre-school age) (Hannula & Lehtinen, 2005). These individual differences in children have been attributed to the existence of a distinct attentional process: the Spontaneous Focusing On Numerosity (SFON). In this context *spontaneous* means a non-prompted action. SFON has been described as an indicator for mathematical development (Hannula & Lehtinen, 2005; Hannula et al., 2010). For example, McMullen and colleagues (2013) investigated the spontaneous recognition of quantitative relations when children were not explicitly guided to notice the mathematical nature of a task. They found that the use of quantitative relations increased with age and that after starting school, children used spontaneous recognition significantly more than before. Moreover, first graders were just as likely to respond based on quantitative relations as they were to respond based on numerosity (McMullen et al., 2013). Altogether, spontaneous focusing on or spontaneous recognising of mathematical aspects of everyday situations has recently drawn the attention of researchers. Extending this approach, the current work investigates the spontaneous recognition and usage of shortcuts. How and when do school children (and later adults) use spontaneous commutativity-based shortcuts?

This question tackles the content core areas of this dissertation. The focus lies in the spontaneous usage of shortcuts and the transfer between them. How could students be supported in spotting and applying shortcuts flexibly? I will bring together cognitive theories about how mathematical concepts and strategies develop. I will furthermore reflect expertise literature to identify differences between the flexibility of experts and novices. Altogether, I will present three potentially supportive contextual factors of spontaneous strategy use by combining these two approaches: I will investigate empirically whether (1) *instructions* has an influence on transfer effects, (2) one commutativity-based shortcut can be triggered by presenting another commutativity-based (*associative influence*), and (3) commutativity is used spontaneously based on/in *estimation* tasks. The contextual factors were subject to experimental manipulations in experiments reported in four journal publications summarized here.

Cognitive fundamentals

It is important to understand the following cognitive fundamentals in the context of mathematical shortcuts: *concept*, *learning and instruction*, *transfer*, *strategy,* and *flexibility*. The cognitive fundamentals *concept*, *learning and instruction* as well as *transfer* merge with each other. Answering the question of how *concepts* are represented helps us to understand how we *learn* what belongs to the concept. How similarity within the *concept* would be determined helps us to understand *transfer* effects. Trying to answer the question under what circumstances *strategies* are used spontaneously and flexibly I will report the literature of strategy change and *flexibility*.

Concepts

Principles can be defined as general rules or regularities that correspond with concepts within a domain, whereas concepts are mental representations constructed by the learner (Prather & Alibali, 2009, 2011). At least three different views exist on how concepts are represented and similarity would be determined, as well as quite a few mixed models (see for a review of all three views Waldmann, 2006). First, in the classic view concepts were represented by rules, which specify required and acceptable conditions for affiliation to the concept. Rule-based representation of mathematical concepts should support the transfer of rules from practiced to less practiced types of arithmetic problems (for instance three-addend-problems rarely practiced in school). According to this view, I would predict that different shortcuts based on the same mathematical principle share a concept. Second, in the prototype view concepts are represented by a number of features or characteristics, which are accepted as true by high likelihood. This means that different shortcuts are represented in one concept if they share specific characteristics. This should lead to an influence of superficial similarity on the transfer of shortcut application. Third, the exemplary view formulates that concepts are represented as an amount of memorised exemplars. So every shortcut as such is represented without extra connective rules, or characteristics. The rule-based view is relevant for strategy change as past work has documented that strategy change can generalize across frequently and infrequently presented instances of the task material (Gaschler, 2009).

Theories of how concepts are represented and operate have largely focused on investigations of concrete entities, which need to be differentiated from abstract concepts such as basic rules of algebra. A meta-analysis of 19 neuroimaging studies showed that abstract concepts need to be differentiated from concrete concepts, because they rely on different systems (Wang, Conder, Blitzer, & Shinkareva, 2010). Theories of grounded cognition also suggest that abstract concepts are represented by distributed neural patterns that reflect their unique content, which is often more situationally complex and temporally extended than that of concrete concepts (Barsalou, Kyle Simmons, Barbey, & Wilson, 2003; Barsalou & Weimer-Hastings, 2005; Barsalou, 1999; Wilson-Mendenhall, Barrett, Simmons, & Barsalou, 2011). Addressing this controversy, Wilson-Mendenhall and colleagues (2013) showed that abstract concepts are represented by distributed neural patterns that reflect their semantic content. Brain regions underlying numerical cognition (e.g., bilateral intraparietal sulcus) are active if semantic content, which is central to arithmetic, is represented (Wilson-Mendenhall et al., 2013). Overall, research of generating concepts in general is essential to understand grouping processes and therefore developing supportive conditions. Note, however, that also the specific content of mental models is relevant in mathematical educational development.

Learning and instruction

Children are taught the commutative, associative, and distributive laws in elementary school. They learn commutativity, as well as the associated shortcuts. Cue learning, a well-investigated cognitive learning model, can be used to explain potentially supportive contextual factors of spontaneous strategy use. Theories of cue learning characterize how people associate cues with particular responses or outcomes (Kruschke, 2001, 2003; Mackintosh, 1975; Rescorla & Wagner, 1972). In cue learning and related models learning is defined as the strengthening of associations between predictive cues. The shifting of attention accelerates learning of new associations, while protecting previously learned associations (Kruschke, 2001, 2003). These associations can be used to trigger mathematical shortcuts. Following the assumption of the classic view of concept representation that shortcuts belonging to the same arithmetic principle are stored together, I suggest that one shortcut might be effective as a trigger for another shortcut based on the same principle.

Although the influence of instruction in learning situations has been investigated thoroughly, the influence of instruction on the flexibility in mathematical strategy use is often neglected. Children do not learn basic skills, such as counting, solely through explicit teaching. Instead, children often engage with their environment spontaneously, which provides rich and robust learning opportunities for the children (Bransford et al., 2006; Ginsburg, Inoue, & Seo, 1999). Instruction is a form of guided discovery that aids learning (Schwartz & Bransford, 1998). Moreover, guided discovery implies that if students explore learning tasks, they are supplemented with some form of instructional guidance (Alfieri, Brooks, Aldrich, & Tenenbaum, 2011). In school the focus should be on self-guided learning, offering hints and using instruction in a more general manner. For example, after instructions on the concept level are given, students generate and transfer correct procedures on their own, even though they were never explicitly instructed in these procedures (Matthews & Rittle-Johnson, 2009).

The *first potentially supportive contextual factor*, which will be investigated in the dissertation, is the influence of instruction on flexible strategy use and transfer. The instruction of a commutativity-based shortcut might promote a transfer to another commutativity-based shortcut based on the same arithmetic principle. The activation of the first shortcut might prime the second one via associative connections between the procedural knowledge of one and the other shortcut or via the association that both shortcut procedures have with the arithmetic concept. On the other hand, the instruction might disturb the flexibility in strategy use. ErEl and Meiran (2011) showed that rule finding (new stimuli, new rule) was impaired by previously applying an instructed (rather than a self-discovered) rule in a discrimination learning procedure. Therefore spontaneous recognition and the application of shortcuts should be investigated without instruction. As a *second potentially supportive contextual factor* I will investigate, whether one commutativity-based shortcut can be triggered by presenting another commutativity-based shortcut offered before.

Transfer

If one supports the application of strategy A, does this promote a generalisation to the strategy B? If so, what characteristics should strategy A and B share to promote this transfer? This question received far too little attention for arithmetic shortcuts. One general definition of transfer is the application of acquired knowledge or learned skills in a new context (Brunstein & Krems, 2006; Frensch & Haider, 2008). The definition of transfer from Ferguson (1954, p. 99) is even wider: 'any effects resulting from repetition, in the ability to perform a specified task, either the same task under different conditions or a different task'.

Six dimensions can be used to structure the domain of transfer (Perkins & Salomon, 1992): (1) positive versus negative transfer, (2) general versus specific transfer, (3) near versus far transfer, (4) vertical versus horizontal transfer, (5) literal versus figural, and (6) low-road versus high-road transfer. Only the first three dimensions are relevant for the transfer from one commutativity-based shortcut to another commutativity-based shortcut. In other words, the transfer from one shortcut to another one might, for example, be facilitating (first dimension). The results will then show whether only a particular aspect of, rather than general attitudes toward the first shortcut affect the transfer to the second shortcut (second dimension) and whether the shortcuts need to be conceptually close or far in order to obtain a transfer effect (third dimension). Only few studies exist that are directly relevant to this topic. They have investigated that during development the integration of the concept increases with more expertise (Haider et al., 2014) and that the components of transfer change qualitatively with practice (Singley & Anderson, 1985). A deeper understanding might lead to a higher usage of principles in the new contexts and therefore to a higher transfer effect, about which, though, we still know too little. First results suggesting a positive transfer from one commutativity-based shortcut to another commutativity-based shortcut (Gaschler, Vaterrodt, Frensch, Eichler, & Haider, 2013) need to be replicated.

Strategy

A strategy is a step-by-step procedure for solving a problem that is non-obligatory and goal directed (Siegler & Jenkins, 1989; Siegler, 1996). This definition has a background in arithmetic strategy research. Note that not all theories in the literature of skill acquisition emphasize changes in processing steps (i.e. steps in calculating arithmetic problems). Some rather see strategy change as the result of avoiding processing steps altogether. For example, the arithmetic problem and the appropriate strategy are stored together so that they are retrieved from memory as a combination (Logan, 1988, 1992). By using unfamiliar problems in the task (e.g. three-addends problems) it is more difficult to directly retrieve the problem-resulting trace. This helps to focus investigations in mathematical education on changes within a repertoire of strategies as well as the flexible application of one or another strategy depending on situational demands.

The issue of how problem solving strategies develop can be subdivided into two parts: strategy discovery and strategy generalisation (Siegler & Jenkins, 1989). The assumption is that if children know one or more strategies for solving an arithmetic problem, than they will most often use the strategy that gives them the highest advantage (e.g., the strategy that is the quickest or easiest). By using this strategy more often, their knowledge about the future application of this strategy increases (strategy generalisation). The key characteristics of whether a strategy can be used in a given situation are accuracy and efficiency. It is very likely that a shortcut leads to an accurate result and saves labour. There might be no difference between strategies and shortcuts (Siegler & Jenkins, 1989).

More or less advanced strategies coexist for young children. Before children are able to calculate sums they are able to count. For example, Lisa has 2 marbles and Tim has 6 marbles. How much do they have together? First, they put them all together and count the whole cluster. Later they will start by 2 and add 6, again through counting. It will save even more time, if they start counting at the higher addend (6 – 7 – 8 like the min-strategy). With time, the children should learn to use more advanced strategies. Siegler and Jenkins (1989) described the use of different more or less advanced strategies as a wave. Flexible strategy change involves the knowledge of strategy efficiency and the development of flexibility is related to transfer and conceptual knowledge growth (Star & Rittle-Johnson, 2008).

Flexibility

Children should develop the skills to spot and apply shortcuts spontaneously and flexibly. Note that the terms ‘flexible’ and ‘adaptive’ in the context of research on individual strategy application in mathematics education are sometimes used as synonyms and sometimes with different meanings. For some researchers the flexible use of strategies means that students are able to choose between different strategies, but the adaptive use of strategies also encompasses the choice of the most appropriate strategy (Heinze, Star, & Verschaffel, 2009). Other researchers do not make this distinction and use the terms interchangeably (Star & Rittle-Johnson, 2008). Adaptive expertise (Verschaffel, Luwel, Torbeyns, & Dooren, 2009) includes to autonomously regulate whether (a) to solve an arithmetic problem in a standard way or to (b) search for and apply a shortcut. That way a person is usually faster and more accurate. Some factors can tip the balance on the flexibility - stability continuum (or exploration vs. exploitation continuum). Experts know when to search for a new shortcut (exploration) and when not (exploitation), but children have to learn how much time and effort they want to spend for the strategy search. In other words, children need to learn how to regulate the dilemma between investing time in calculating in a standard way or to search for and change towards alternative strategies. Teachers can only help students to calibrate the balance between flexibility and stability (or exploration vs. exploitation). Altogether, a key learning outcome in mathematical arithmetic is the development of flexible knowledge, in which students know multiple strategies and can spot and apply them adaptively to a range of situations. Students need to be supported to reach this aim during the development of mathematical abilities.

The development of mathematical abilities

The development of mathematical concepts

As I already mentioned there is evidence that concrete concepts need to be differentiated from abstract concepts. Abstract concepts are crucial for investigating the application of shortcuts. Resnick (1992) investigated how informal knowledge is transformed into formal mathematics. The results showed that by discovering physical objects and later the combination of physical and numeral entities a *sense of number* develops. Researchers have noted a difference between *conceptual knowledge* as implicit or explicit understanding of the principles and *procedural knowledge* as the ability to execute action sequences to solve problems. Baroody and Gannon (1984) proposed a four-stage model on mathematical development, in which procedural knowledge is the basis for mathematical development. After counting, conceptual knowledge develops and in the last stage both procedural and conceptual knowledge are integrated (Baroody & Gannon, 1984). However, it is debateable whether conceptual or procedural knowledge develop first (Rittle-Johnson et al., 2001). Based on procedural- or conceptual-first models, different interventions and teaching strategies developed. Note that children differ in their prior knowledge and amount of experience with the relevant concepts before the target procedure is taught in school. It can also depend on the domain of whether children have prior procedural or conceptual knowledge (Rittle-Johnson et al., 2001). Procedural and conceptual knowledge are often linked. Improving children's knowledge of one type can lead to improvements in the other type. The advantage of an iterative model is a reduced importance of the question of what came first. The iterative model of Rittle-Johnson and colleagues (2001) focuses on the mechanisms underlying the influence of each type of knowledge upon the other. For example, problem-solving experience and feedback, conceptual knowledge, and representational support enhance problem representation. Additionally, problem representation is a mediator of the relationship between prior conceptual knowledge and improved procedural knowledge. However, Rittle-Johnson and colleagues (2001) assumed that improved choices among competing procedures may be mechanisms to use prior to conceptual knowledge to improve procedural knowledge. Otherwise they also found evidence that procedural knowledge lead to an improvement of conceptual knowledge. This link between conceptual knowledge and procedural knowledge in problem representation is bidirectional. Children should have more available resources for observing relationships between problems and reflecting the concepts underlying them by needing fewer mental resources to solve mathematical problems (Rittle-Johnson et al., 2001). Conceptual knowledge is also important to guide attention to task relevant information in order to solve problems and to transfer knowledge from one task domain to another flexibly (Anderson & Schunn, 2000; Gentner & Toupin, 1986; Haider & Frensch, 1996; Koedinger & Anderson, 1990).

Altogether, there is a strong bidirectional relationship between conceptual and procedural knowledge. Both types of knowledge improve each other during the development of mathematical concepts. Consequently, knowledge integration leads to a transfer between procedurally different shortcuts that are based on the same mathematical principle and, therefore, are both likely to be associated to the respective conceptual knowledge. Following this assumption, I will predict that it is possible to trigger a commutativity-based shortcut by presenting another ‘easier to find’ commutativity-based shortcut before.

The focus of the current work was on strategy use, which might be tied to specific problem types and is not widely generalizable. Thus, the aim was to assess procedural knowledge. In other work with similar task material procedural and conceptual knowledge was assessed (Haider et al., 2014; Hansen et al., submitted). The aim of this dissertation is not to examine the development of arithmetic strategies directly, but rather to examine different contextual factors of spontaneous and flexible strategy use (of strategies that are already known). Still, we need to understand mathematical development to interpret the results properly.

A wealth of research has confirmed that children with better conceptual knowledge are also able to solve arithmetic problems better and have more varied strategies in their repertoire than children with weaker conceptual knowledge do (Alibali, 2005; Baroody, Feil, & Johnson, 2007; Canobi, 2005, 2009; Siegler & Stern, 1998; Siegler & Svetina, 2006). Thus, increasing the experience in a domain will lead to a more integrated mathematical concept and therefore to a more flexible use of different strategies.

Expertise

Expertise is domain specific and has various manifestations. In recent years, research in primary school arithmetic started to tackle this issue in a domain in which everyone should acquire elaborate knowledge. Adaptive expertise (Verschaffel et al., 2009) has been linked to autonomously regulating whether (a) to solve an arithmetic problem in a standard way or to (b) search for and apply a shortcut. As implied by these authors, children should develop the skills necessary to spot and apply shortcuts spontaneously and flexibly. It is not sufficient if they can apply a shortcut when explicitly told to do so. An expert differs from a novice in spontaneously recognizing where knowledge can be applied to simplify the task and in transferring his/her knowledge to new situations.

Expertise is defined as consistent superior performance compared to novices and compared to other domains (Ericsson & Lehmann, 1996). There is an extensive discussion about the influence of expertise on flexibility, with two contradictory perspectives. On the one hand, research on creativity and skill acquisition has shown that more knowledge can make people less flexible (i.e., Logan, 1988; Luchins, 1942). On the other hand, research on expertise suggests that the thought patterns of experts are more flexible and creative (for a short review see Bilalić, McLeod, & Gobet, 2008). Bilalić and colleagues (2008) investigated chess experts and found both flexibility and inflexibility in experts depending on the expertise level and the problem difficulty. Here I focus on the domain of mathematics. Mathematics students at University level use significantly larger numbers of appropriate strategies than adults with less expertise do (Dowker, Flood, Griffiths, Harriss, & Hook, 1996). With further experience students become increasingly able to generate rapid adequate actions with less and less effort (Ericsson, 2008). Hatano (1988) distinguishes routine from adaptive expertise. Routine is defined as using a procedure quickly and accurately without necessarily understanding it, whereas adaptive expertise is the ability to apply learned meaningful procedures flexibly and creatively. This definition of expertise fits into the context of mathematical concepts very well, because the construct of *understanding* in adaptive expertise means the integration of procedural and conceptual knowledge (Verschaffel et al., 2009).

Approximate arithmetic

Becoming an expert is the end of a long journey, but when does it start? A wealth of research has confirmed that toddlers develop an informal understanding of relations between objects in the real world (e.g. Baroody & Gannon, 1984; Resnick, 1992). This understanding deepens when they enter school. For example, Sherman and Bisanz (2009) showed that working with non-symbolic material can encourage subsequent strategy use in symbolic equivalence problems. However, the reversed order (starting with symbolic) did not affect the performance in the later non-symbolic problems. An additional line of studies focuses on children's ability in approximate calculation (e.g., Barth et al., 2006; Barth, Mont, Lipton, & Spelke, 2005; Gilmore, McCarthy, & Spelke, 2007, 2010). Children can use the representation of approximate numbers to perform addition and subtraction even prior to learning arithmetic in school (Gilmore et al., 2007). There is a relationship between their performance of non-symbolic approximate arithmetic and children's success in mathematics at the beginning of learning (Barth et al., 2006; Gilmore et al., 2010).

Exploring specific mathematical problems or presenting non-symbolic problems might help students to understand abstract mathematical principles better. This might also be true for approximate calculation (Gilmore et al., 2007, 2010; Gilmore & Spelke, 2008). Hansen and colleagues (submitted) tested whether children would benefit from an estimation task involving commutativity-based arithmetic problems on later arithmetic problems. They confirmed the assumption that symbolic estimation increases the spontaneous spotting and application of commutativity-based shortcuts in a later arithmetic task. I try to replicate this finding and therefore the *third potentially supportive contextual factor* of spontaneous strategy use is approximate arithmetic. I was interested in whether/why commutativity is used spontaneously based on estimation tasks and whether commutativity is used spontaneously in estimation tasks. Shortcuts that entail comparing addends across subsequent addition problems require unusually long eye movements. The influence of flexibility in visual patterns on spontaneously recognising shortcuts has not been investigated yet. It is debatable if estimation as such raises the chance that a child spots and applies the shortcut. Another possible explanation is gaze patterns - triggered by an estimation task. These gaze patterns might still be present during the estimation task when later faced with an addition task.

There might be other estimation situations, in which commutativity is used spontaneously, for example graph processing. Graphs can be attributed to a particularly short link of number representations, especially to analogy representations of numerosity (Gallistel & Gelman, 1992, 2000). Research on the estimation of quantities from bar graphs suggests that perceptual properties are the most important determinants of graph processing (Meyer, Taieb, & Flascher, 1997). I have adopted a version of graph processing focusing on the spontaneous use of commutativity-based shortcuts. Combining both assumptions (Gallistel & Gelman, 1992; Meyer et al., 1997), my colleagues and I investigated whether commutativity benefits from bar graphs can be differentiated from perceptual effects, such as mirror symmetry and pattern repetition.

I have discussed what may influence the spontaneous usage of shortcuts in this section. I will investigate three potentially supportive contextual factors of spontaneous strategy use by combining different approaches from literature: (1) the influence of *instruction* on transfer effects, (2) *associative influence* or otherwise triggering one commutativity-based shortcut by presenting another commutativity-based shortcut before, and (3) commutativity used spontaneously based on/in *estimation* tasks.

Measuring commutativity effects in arithmetic

The review of Prather and Alibali (2009) on measuring what learners know about mathematical principles differentiates between five types of knowledge assessment: the application of procedure, evaluation of procedures, justification of procedures, evaluation of examples, and explicit recognition. It is difficult to measure commutativity effects in arithmetic performance, because there are differences between competence and performance depending on the type of knowledge assessment used. I will present the three main measurements of commutativity, which are used very often in literature.

First, it is possible to measure commutativity through the usage of counting strategies. Counting strategies exist, which are a direct predecessor of commutativity-based shortcuts. For example, if children want to add up two numbers they will start counting from the higher addend. This means, they learn to switch the addends (counting-on-larger, counting-all-larger or sometimes also called min-strategy). This measurement should only be used with very young children, because when children use specific counting strategies it is not clear whether they use the commutativity principle (Prather & Alibali, 2009).

Secondly, it is possible to use verbal report for the evaluation or justification of a procedure. For example, students are presented with an arithmetic problem and afterwards they will be asked to describe how they solved the problem. Another possibility is to ask children whether it is allowed to solve the problem in this or that way. Usage of verbal report is frequently reported in the literature (Cowan & Renton, 1996). The problem with self-/verbal-reports is that they might trigger the concept. If the children were instructed to further explain their strategies, they might receive a hint about the existence of commutative problems in the tasks. I wanted to evaluate the spontaneous use of commutativity-based shortcuts. Therefore I did not use verbal report.

Thirdly, it is possible to use the iteration of problems in order to measure commutativity as the application of a procedure (e.g., 8 + 5 + 7 = ?, and afterwards 5 + 7 + 8 = ?) (Prather & Alibali, 2009). These problems could vary in the examined paradigm (e.g., object-like, verbal, symbolic task material). The iteration of problems means presenting an addition problem followed by the presentation of a problem with the same addends in a different order. We used this possibility as one possible shortcut. For example, we offered 8 + 5 + 7 = ?, and then 5 + 7 + 8 = ?. Here, children could use a commutativity-based shortcut across problems. We called it the "Addends-compare strategy". The commutativity principle enables students to change the order of addends flexibly and therefore avoid calculating the second problem. Additionally, we offered a commutativity-based shortcut within one problem. For instance, given the problem 4 + 7 + 6, it might be easier to calculate (6 + 4) + 7 (6 + 4 adds up to 10 which makes it easy to finally add 7). We called it the "Ten-strategy".

Eye tracking or reliance on aggregate measures from paper-and-pencil versions might both be useful approaches. Combining both approaches - eye tracking in a laboratory and paper-and-pencil in the school setting – gives the full picture.

Methods

Samples and procedures

Five experiments with a total of 399 elementary school children and 100 adults were conducted. The exact descriptions of the samples are reported in the journal articles. The exact procedures of the five experiments are described in the journal articles, as well. We conducted the experiments either as a paper-and-pencil test in separate booklets or on a computer screen. Using paper-and-pencil assessment in classrooms has advantages over laboratory testing: first, the study has a high ecological validity, because the children are in their familiar environment. Second, we do not trigger strategy use because it is not obvious to the students what is being measured. However, using laboratory assessment had advantages over classroom testing: there are fewer distractions in the laboratory and we were able to measure the solution times per problem more precisely and could also obtain eye tracking indicators of strategy use. Therefore I combined both approaches – eye tracking in a laboratory and paper-and-pencil in the school setting. For instance, eye tracking can help to figure out whether increased time demands after a change in shortcut option reflects prolonged solution times or if it reflects a mixture of prolonged solution times plus time invested in search for alternative shortcut options.

In the classroom testing different types of problems were presented in separate booklets. Students were instructed to solve the problems as quickly and as accurately as possible. Additionally, they were informed that it would be almost impossible to solve all problems during the period of time given for each booklet. Assessing eye tracking in the laboratory was a single-testing situation. After calibration, children were presented with the material on the screen and had to solve the problems after each other. The children verbalized the result of a problem and the experimenter entered the answer and moved the cursor to the next problem, a line above.

Materials

We used very similar materials for most of the experiments. I will first describe the material for measuring commutativity and then I will describe the material for inducing and measuring estimation.

Arithmetic task including commutativity

The materials we used to test the specific shortcuts are depicted in Table 1 (addends-compare and ten-strategy). Prior studies (Gaschler et al., 2013; Haider et al., 2014), that tested whether the two commutativity-based shortcuts addends-compare and ten-strategy would be applied, used the same material in paper-and-pencil assessment and eye tracking. Each problem was presented in one line and consisted of three different addends. Each page in the booklet with the addends-compare strategy problems contained two pairs of commutative problems (one problem and the repetition with different order of addends) and two control problems. In the corresponding baseline booklet, no such addends-compare strategy problems were used. Instead, students received pairs of control problems that had the same result but were composed of different addends. In the ten-strategy problem booklets, each problem offered the possibility to use a commutativity-based shortcut (i.e., the ten-strategy). There was a corresponding baseline booklet as well.

Dependent variables

In the current experiments my colleagues and I mainly measured how fast and how accurately children solved addition problems. We measured the benefit of using a shortcut as our main dependent variable. In the paper-and-pencil studies, the benefit is computed by subtracting the number of problems solved in the booklet that allowed for the commutativity-based shortcut from the number of problems solved in the booklet that did not allow for the commutativity-based shortcut (baseline problems). We made sure that the time provided per booklet was not sufficient to solve all problems, so that we could use the number of problems solved as a dependent variable.

We measured the solving times more precisely in the laboratory experiments than in the classroom setting. In the eye movement experiments we also measured the percent of fixation on a specific point of interest and the distribution of saccade distances. For a more detailed analysis, we differentiated between horizontal and vertical saccades.

Table 1
Examples of the first five problems of each problem type (*ten-strategy/addends-compare strategy problems* and *baseline*) in the parallel sets A and B. The results printed in italics had to be filled in by the participants.

Set A		*Set B*	
Ten-strategy	*Baseline*	*Ten-strategy*	*Baseline*
4+5+6=*15*	4+3+8=*15*	6+5+4=*15*	8+4+3=*15*
3+2+7=*12*	3+5+4=*12*	7+2+3=*12*	3+4+5=*12*
5+6+5=*16*	8+5+3=*16*	5+9+5=*19*	8+3+5=*16*
7+4+3=*14*	2+5+7=*14*	3+4+7=*14*	5+2+7=*14*
2+7+8=*17*	9+3+5=*17*	8+7+2=*17*	5+3+9=*17*
Set A		*Set B*	
Addends-compare	*Baseline*	*Addends-compare*	*Baseline*
3+5+4=*12*	5+3+4=*12*	4+3+5=*12*	4+5+3=*12*
4+9+8=*21*	8+9+4=*21*	5+7+9=*21*	8+4+9=*21*
4+8+9=*21*	6+7+8=*21*	5+9+7=*21*	7+8+6=*21*
6+2+5=*13*	5+2+6=*13*	2+6+5=*13*	2+5+6=*13*
9+7+2=*18*	2+7+9=*18*	4+5+9=*18*	7+9+2=*18*
2+9+7=*18*	9+4+5=*18*	5+4+9=*18*	5+9+4=*18*

Estimation task

In two experiments we manipulated or used estimation. Depending on the research question, we used two different estimation tasks. In Journal Article 3 elementary school children had to estimate which of two characters (Tim or Lisa) owned more marbles or if both owned the same amount of marbles. Therefore we used material from Hansen et al. (submitted) in an adaptive form (Figure 1). The quantities of marbles were presented at right and left edge of the presentation frame in order to induce long-range eye movements. In the control group the marbles were presented centrally (for more details see Journal Article 3).

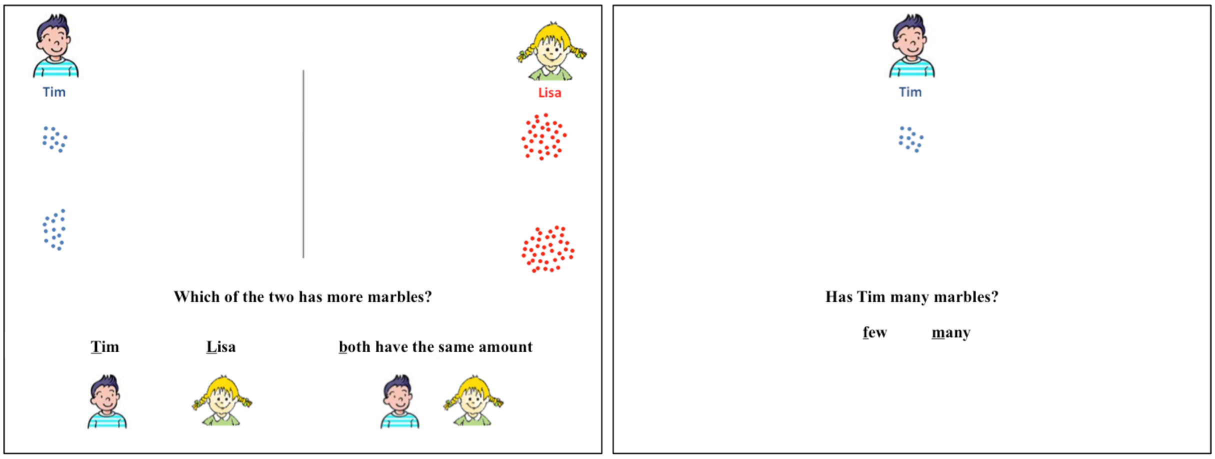
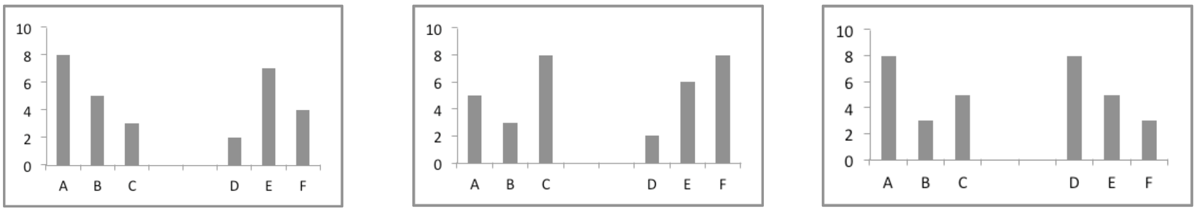


Figure 1. Example of the marbles estimation task for the long-range eye movements (left) and the centred group without long-range eye movements (right).

In Journal Article 4 adults were presented with a different estimation task. They were shortly presented with a bar graph and then had to estimate the sum of three bars on the right side and compare it with the sum of the three bars on the left side of the bar graph. The task was to decide whether or not the sum of the left triplet equals the sum of right triplet. These sums were taken from three conditions (Figure 2) (1) unequal sum, (2) equal sum, but composed of different addends, and (3) equal sum composed of identical addends. In the latter condition the addends-compare strategy could be used (e.g. left triple: 8, 3, 5 right triple: 8, 5, 3).



Unequal sum Equal sum Equal addends

Figure 2. Example of the bar graph estimation task for the condition unequal sum (left), the equal sum (center) and equal addends (right).

Hypotheses

This work will focus on the following three potentially supportive contextual factors of spontaneous strategy use while examining the spontaneous usage of mathematical shortcuts by using the commutativity principle. I will empirically investigate (1) the influence of *instruction* on transfer effects, (2) the *associative influence* or otherwise triggering one commutativity-based shortcut by presenting another commutativity-based shortcut before, (3) *estimation* and whether commutativity used spontaneously based on/in estimation tasks.

The first goal of the current study was to test whether an explicit instruction about the commutativity-based addends-compare strategy increases the usage of a different commutativity-based shortcut (i.e., the ten-strategy). Several outcomes are possible: A direct use of the first commutativity-based shortcut might activate the concept, which then supports the second one. Or the instruction might hinder the flexible usage of knowledge about mathematical principles. Based on the theoretical considerations above, I state the contrasting hypotheses:

1) The influence of instruction on transfer effects

a. Instruction promotes the transfer of commutativity-based shortcuts: the group that received an instruction of the addends-compare strategy will later on show more benefit of the ten-strategy than the group without instruction.

b. Instruction hinders the transfer of commutativity-based shortcuts: the group without instruction of the addends-compare strategy will later on show more benefit of the ten-strategy than group that received instruction.

The next potentially supportive contextual factor for spontaneous and flexible use of appropriate shortcuts in addition problems might be *associative influence* or otherwise triggering one commutativity-based shortcut by presenting another commutativity-based before. If one allows the application of strategy A and it promotes generalisation to the strategy B the characteristics which strategy A and B share can be divers. Conceptual knowledge concerning commutativity could play a role in transfer. Furthermore, different commutativity-based shortcuts are linked: applying commutativity within a task (ten-strategy: $6 + 7 + 4 = (6 + 4) + 7$) is conceptually linked to applying the mathematical principle between

two tasks (e.g. 5 + 8 + 3 = followed by
8 + 5 + 3 =). Otherwise, starting a series of arithmetic problems with problems that contain *any* options for shortcut use (e.g. inversion) may motivate the search for other shortcut-options in later arithmetic problems. The inversion shortcut (e.g., 9 + 2 − 2) does not belong to the commutativity principle. Based on the theoretical considerations above, I state two hypotheses:

2) Triggering one commutativity-based shortcut by presenting another commutativity–based shortcut before

a. The transfer within the mathematical concept of commutativity: the condition, in which the children started with the ten-strategy will show more benefit of the addends-compare strategy later than the other two conditions will (baseline and inversion).

b. Concept-independent transfer - a pure motivational effect: the conditions, in which the children started with a possibility to use a shortcut (inversion, ten-strategy) will show more benefit in the addends-compare strategy later than the baseline condition will.

Pre-schoolers already develop an informal understanding of relationships between objects in the real world (e.g. Baroody & Gannon, 1984; Resnick, 1992). There is a relationship between the performance of non-symbolic approximate arithmetic and children's success in mathematics, at the beginning of learning (Gilmore et al., 2010). The final potentially supportive contextual factor for spontaneous and flexible use of appropriate shortcuts in addition problems I address in this dissertation is estimation. As argued above, estimation has a supportive effect on flexibility (Hansen et al., submitted). However, this research neglected the possibility that induced long-range eye movements may have provoked the flexibility. Therefore, the first aspect may account for the influence of the (non)flexible eye movements during an estimation task.

Furthermore, I explored whether commutativity is used spontaneously in estimation tasks. I investigated whether there might be a graphical equivalent of commutativity-based shortcuts in arithmetic. As opposed to the previously used estimation task, I used bar graphs and tested whether adults used commutativity in the processing of briefly presented bar graphs spontanously. Based on the theoretical considerations above, I state the hypotheses:

3.1) Commutativity is used spontaneously based on estimation tasks

a. Fixation patterns account for the transfer from estimation to arithmetic tasks: the group that had to do long-range eye movements will later on show more benefit of the addends-compare strategy than the group that focused on the centre.
b. Fixation patterns do not account for the transfer from estimation to arithmetic tasks: the group that focused on the centre and therefore had no long-range eye movements will show similar benefit of the addends-compare strategy.

3.2) Commutativity is used spontaneously in estimation tasks

a. Commutativity influences the estimation of graphically represented numerical quantities: in the condition in which the commutativity-based shortcut could be used the reaction will be more accurate and faster compared to the baseline condition.
b. Commutativity does not influence the estimation of graphically represented numerical quantities: in the condition in which the commutativity-based shortcut could be used the reaction will be similar to the baseline condition.

Results

Before I present the results of testing the hypotheses I will present the results of an eye tracking study, which aimed to test the differences of the eye movement pattern of the specific shortcuts (addends-compare and ten-strategy). On the one hand these results serve as a manipulation check of the two different shortcuts and the material we used for measuring them. On the other hand the eye movement results serve as an argument for another experiment, which will be discussed later.

Eye movements

My colleagues and I conducted an experiment to test eye movements while solving arithmetic problems that offer the possibility to use commutativity-based shortcuts (Journal Article 2, Experiment 1).

The results showed that the ten-strategy and the addends-compare strategy could be identified by specific fixation patterns. For example, if one uses the ten-strategy (6 + 7 + 4 = (6 + 4) + 7), the fixations might be distributed according to the middle vs. outer numbers of the three-addends problems. We found that the percentage of fixations on the middle number increased over time during the ten-strategy problems, as students presumably discovered the structure of the problems. If one uses the addends-compare strategy, one needs to look back and compare the addends of the last problem with the addends of the current problem. Students were fixating ahead on problems preceding the addends-compare problems and fixating back, once they discovered a set of identical addends.

In addition to *identifying* the eye movement patterns that are specific for the shortcuts, we also found a significant correlation between these patterns and the solving time benefit. The fixation on the middle digit, which is specific for using the ten-strategy, increased over time. The ten-strategy specific fixation correlated significantly with the time benefit in addends-compare strategy problems.

The influence of instruction on transfer effects (Hypothesis 1)

We tested whether instructing the addends-compare shortcut would help fourth graders to transfer this shortcut to a different class of problems that require a different shortcut, but are based on the same mathematical principle, spontaneously. We checked if the explicit instruction to use the addends-compare strategy increased its application for the direct instruction condition. The fourth graders benefitted from the instruction significantly and, therefore, used the addends-compare strategy. Then, we checked the influence of the instruction of the ten-strategy (transfer effect). The fourth grader's benefit for the ten-strategy seemed to be even smaller in the instructed compared to the non-instructed group on a descriptive level. The results indicate that both hypothesis need to be rejected. The instruction neither promoted nor hindered the transfer of commutativity-based shortcuts significantly.

The results suggest that the direct instruction of one commutativity-based shortcut does not affect the spontaneous usage of a different shortcut based on the same mathematical principle positively. Rather, the tendency towards the opposite seems to be the case. The instruction of the addends-compare strategy seems to be detrimental to spotting and applying the ten-strategy flexibly.

Triggering one commutativity-based shortcut by presenting another commutativity-based shortcut before (Hypothesis 2)

We tested whether the spontaneous shortcut usage of the ten-strategy triggers knowledge about the other commutativity-based shortcut (addends-compare). The second graders, who started with the ten-strategy problems, showed a significant benefit in the addends-compare strategy problems compared to baseline problems. This result supports the Hypothesis 2a concerning the transfer within the mathematical concept of commutativity. Second graders, who started with inversion problems ($a + b - b = ?$), did not show such a benefit. This result rejected the hypothesis of concept-free transfer - a pure motivational effect.

We did not find differences between the conditions for the third graders (starting with ten-strategy; inversion or baseline). Each of the three groups starting with different shortcut options used the addends-compare strategy and ten-strategy later. Their benefit was significant. They were faster in the addends-compare strategy problems compared to the baseline problems. They used the commutativity-based shortcut independently and apart from the task before. Thus, the results of the third graders neither supported the concept-specific nor the motivational account of transfer.

Commutativity is used spontaneously based on/in estimation tasks (Hypotheses 3.1, 3.2)

We tested whether fixation patterns account for transfer from estimation to arithmetic tasks with the marbles estimation task (Figure 1), and we focused on saccade distances in the analysis of the eye movements. The results showed different eye movement patterns in the arithmetic task presented afterwards. The results showed that inducing long-range eye movements does not influence the usage of commutativity-based shortcuts in arithmetic. The result supported the Hypothesis 3.1b that states that fixation patterns do not account for the transfer from estimation to arithmetic tasks.

While shortcut search and application is reflected in fixation patterns, we did not obtain evidence for the reverse influence. Inducing long-range eye movements did not lead to a higher shortcut usage (addends-compare strategy).

We tested whether commutativity influences the estimation of graphically represented numerical quantities with the bar graph estimation task (Figure 2). We obtained a commutativity benefit for bar graphs that can be differentiated from perceptual effects, such as mirror symmetry and pattern repetition. Accuracy was higher and the reaction time was lower when the commutativity-based shortcut could be used, compared to the baseline condition. This supports Hypothesis 3.2a that commutativity influences the estimation of graphically represented numerical quantities. Overall, the results indicate that there might be a graphical equivalent of commutativity-based shortcuts in arithmetic.

Discussion

A total of five experiments in classroom and laboratory settings with different samples (children and adults) yield the following results: The usage of the addends-compare and ten-strategy is reflected in different *eye movements*. In addition identifying the eye movement patterns that are specific for the shortcuts, we also found a significant correlation between these patterns and the time benefit. After direct *instruction* of the addends-compare strategy we observed no transfer to a different commutativity-based shortcut. For the spontaneous usage of shortcuts we obtained different results for *triggering one commutativity-based shortcut* in second and third graders. In the second grade there was no transfer from a shortcut that is not based on commutativity (inversion-group) to one that is. Second graders could be supported in using the addends-compare strategy if the easier ten-strategy was presented before, whereas third graders used the addends-compare strategy in every condition and did not show extra benefits from supportive context factors. While shortcut search and application is reflected in fixation patterns, we did not obtain evidence for the reverse influence. Inducing long-range eye movements in an *estimation task* did not lead to higher shortcut usage (addends-compare strategy) in a later arithmetic task. With an adult sample we showed that commutativity is used spontaneously in a graphical estimation tasks.

The eye tracking data (Journal Article 2, Experiment 1) coincides with the interpretation of search processes starting once one shortcut no longer applies. Moreover, we found anticipatory behaviour, which supports the assumption of an active search process. I drew the following conclusions from the correlation between the ten-strategy specific fixation and the time benefit in addends-compare strategy problems: first, it validates that by measuring solution time benefits I could operationalize the application of the arithmetic strategy. Second, it is consistent with the view that the two different commutativity-based shortcut strategies might be related via the concept of commutativity. I could show that the time needed for the search process needs to be taken into account when interpreting solving times with this experiment.

The goal of Journal Article 1 was to test whether an explicit instruction about a commutativity-based shortcut would increase the usage of a different commutativity-based shortcut. The results comply with research about rule-governance. Participants, who were given the rule as an instruction, consistently showed greater difficulty to adjust to the new conditions than the non-instructed participants (Hayes, Brownstein, Zettle, Rosenfarb, & Korn, 1986; Törneke, Luciano, & Salas, 2008). Moreover, rule finding (new stimuli, new rule) was impaired by previously applying an instructed (rather than a self-discovered) rule in a discrimination learning procedure (ErEl & Meiran, 2011).

We found that the transfer could not be supported with instruction. This did not fit into the emerging consensus that people learn best through some form of guided discovery that combines exploration and instruction (e.g. Alfieri et al., 2011; Hmelo-Silver, 2007; Lorch Jr. et al., 2010; Mayer, 2004). But our results are in line with work reporting less transfer in a more specific instruction condition than in a more general one (Kaminski, Sloutsky, & Heckler, 2008).

The instruction of a commutativity-based shortcut did not affect the spontaneous usage of a different shortcut based on the same mathematical principle positively. Rather, the opposite seemed to be the case. Gaschler and colleagues (2013) previously found that the spontaneous (non-instructed) usage of the addends-compare strategy and the ten-strategy is positively correlated in the third grade. Because of this finding and the educational standard, which ensures the flexible use of the basic rules of algebra, we measured the spontaneous strategy use in the following experiment. This spontaneous strategy use is crucial in investigating the transfer from one commutativity-based shortcut to another commutativity-based shortcut. We offered the easy-to-find commutativity-based shortcut to investigate the transfer to another commutativity-based shortcut. We cannot ensure that the first strategy is recognized without instruction.

Second graders in Journal Article 2 Experiment 2 seemed to have benefitted from an easy-to-find commutativity-based shortcut, wherever third graders are more experienced and did not seem to benefit from such extra scaffolding. With further experience, students become increasingly able to rapidly generate adequate actions with less and less effort (Ericsson, 2008). Differences in mathematical abilities between second and third graders are mirrored in the functional changes of the brain, which coincides with our findings. Brain response and connectivity relating to an arithmetic task change significantly within the narrow 1-year interval (age: 7/8 – 8/9 years) (Rosenberg-Lee, Barth, & Menon, 2011).

In the case of commutativity, triggering one commutativity-based shortcut by presenting another commutativity-based shortcut beforehand might support contextual factors for spontaneous shortcut use in primary school arithmetic. The additional conceptual link between the two different strategies may be the reason for the transfer. Conceptual knowledge can guide people's choices concerning alternative procedures (see Crowley, Shrager, & Siegler, 1997; Shrager & Siegler, 1998). Spontaneous usage of one shortcut seemed to transfer to the other shortcut, so that transfer might be concept-specific.

Besides the assumption of transfer via the concept, another explanation might be the motivational factor. A possible assumption might be that after having experienced that task, processing can be simplified by a shortcut, so that one is more apt to search for and apply *any* shortcut. This could lead to a transfer based on the motivation to simply search for shortcuts. Our results (Journal Article 2, Experiment 2) did not support this assumption. Offering problems with an easy-to-find shortcut option inversion (not commutativity-based shortcut) or ten-strategy (commutativity-based shortcut) only indicated a transfer from the ten-strategy to the other commutativity-based shortcut (addends-compare strategy). Rather than searching for any shortcut option after having spontaneously discovered one in a first problem, participants might specifically search for shortcut options that are based on the same mathematical principle (Prather & Alibali, 2009).

In order to test whether using commutativity spontaneously based on estimation tasks is a potentially supportive contextual factor, I investigated, in Journal Article 3, the influence of long-range eye movement patterns during an estimation task. The results coincides with findings that demonstrated that estimation per se (rather than eye movements induced by estimation) might positively influence exact calculation (Gilmore et al., 2007, 2010). However, we found different fixation patterns in the arithmetic task. This result suggests that estimation problems can indeed influence fixation patterns in a later arithmetic task. Inducing long-range eye movements did not lead to higher shortcut usage (addends-compare strategy). This result is also in line with top-down accounts of strategy change: fixation patterns reflect rather than elicit strategy change (cf. Haider & Frensch, 1999). Hansen and colleagues (submitted) confirmed the assumption that symbolic estimation increased the spontaneous spotting and application of commutativity-based shortcuts in a later arithmetic task. Children did not benefit from the estimation task in spotting and applying shortcuts in later arithmetic problems, because long-range eye movements are helpful.

I also considered whether commutativity is used spontaneously in estimation tasks, especially in bar graph processing (Journal Article 4). The results coincide with findings showing that perceptual effects, which appear in number representation, could also be found in graph comprehension (Fischer, Dewulf, & Hill, 2005). Designing optimal graphs can benefit from research into number representations. Additionally, we found perceptual effects such as mirror symmetry and pattern repetition. Prior knowledge, as well as graph design, has an impact on the information processing regarding the presentation of scientific results in graphs (Meyer et al., 1997). Summed up, there might be a graphical equivalent of commutativity-based shortcuts that are spontaneously applied when addition problems are presented in numbers.

Theoretical and practical implications

Children should develop the skills necessary to spot and apply shortcuts spontaneously and flexibly (Verschaffel et al., 2009). The different results between second and third graders confirm the relationship between knowledge development and transfer. How, precisely, development has an impact remains unclear. We found out that older children (third graders) are faster in calculating in general, and that they use and search shortcuts more often. Increased use of correct procedures (and decreasing use of incorrect procedures in turn) is a crucial aspect of improved procedural knowledge (Lemaire & Siegler, 1995; Rittle-Johnson & Siegler, 1999). Even if children behave as if they do possess procedural knowledge about commutativity, their learning process has not come to an end. It still progresses until a well-integrated, abstract representation of a mathematical principle like the commutativity principle is reached (Haider et al., 2014). Conceptual and procedural knowledge are both important for supporting procedural flexibility (Schneider, Rittle-Johnson, & Star, 2011). For this reason, helping students to develop well-integrated knowledge should be one of the most important tasks in education (see e.g., Geary et al., 2008; Prather & Alibali, 2009).

The aim for mathematical development is an integrated concept and is indicated by the spontaneous use of the appropriate strategy for the given problem. I suggest that a milestone lies in balancing (a) spending time on processing arithmetic problems in a well-established manner and (b) searching for potential shortcut options. I showed that explicitly introducing a certain strategy does not seem to be an efficient way to help students to spot and apply shortcut options flexibly. One option of combining instruction and spontaneous flexibility could consist of a more general instruction to encourage the metacognition. A metacognitive system allows the discovery of new strategies by detecting ways in which existing knowledge can be recombined into new strategies (Robinson & LeFevre, 2012). I showed that the conceptual link between strategies might be supportive. Novices, who compared procedures immediately, were more flexible problem solvers. Greater flexibility was associated with greater knowledge of conceptual and procedural knowledge (Rittle-Johnson, Star, & Durkin, 2012). These metacognitions should be trained more intensely because greater metacognitive knowledge has been observed among flexible, gifted students who worked fast (Dover & Shore, 1991). Students are limited in their ability to self-monitor and self-evaluate their problem solving at the individual level. The development of metacognition should be taught in school, because the social level of metacognition is important to potentially overcome individual limitations through feedback and criticism from others (Kim, Park, Moore, & Varma, 2013).

Limitation

As opposed to these studies (Haider et al., 2014; Hansen et al, submitted), I did not assess conceptual knowledge. In many cases the effects of spontaneously using a shortcut were small and the variability across participants was large. This is to be expected for two reasons. First, the difference between competence and performance should be taken into account (i.e., principle knowledge vs. application). Larger estimates of both procedural and conceptual knowledge have been obtained when knowledge was probed more directly (Prather & Alibali, 2009), but the focus of this work is on spontaneous usage. Almost all children would express knowledge (at least partly) about the commutativity principle, if you ask them directly or test them otherwise directly (Baroody, Ginsburg, & Waxman, 1983; Canobi, Reeve, & Pattison, 2002, 2003). But only a few apply arithmetic shortcuts spontaneously (Gaschler et al., 2013; Klein & Bisanz, 2000; Stern, 1992). Second, we measured the use of shortcuts in the classroom, which is a group setting. Group settings are prone to more distractions.

The result of instruction hindering flexibility is not generalizable. Only, the specific variant of the instruction (directly explain the principle by using one example and ask children to look for that shortcut and use it) didn´t support flexibility. Other instructions, for example, that are more general or a call for flexibility might, as well, work.

The connection between psychological data and educational practice has often been difficult to forge (Newcombe et al., 2009). This dissertation cannot bridge the gap between theoretical understanding of basic cognitive processes and practise in classrooms. However, I combined different approaches, because I also conducted an experiment in a psychological laboratory.

One potential limitation pertains to the result that the transfer between shortcuts might be concept specific. In order to gain first insights to the contextual factors of spontaneous and flexible strategy use I used the arithmetic principle commutativity and a few possible shortcuts. First of all, other combinations with other mathematical principles (e.g. inversion) might show different results for a transfer. Future research should use other combinations to investigate whether these findings are broad or very specific. Secondly, it is interesting to note how strong the influence of similarity at the concept level and how strong the influence of superficial similarity might be. I have not systematically varied superficial similarity. In principle, however, this would be feasible. Future work could consider all the possible order combinations to answer the question of whether procedural similarity (looking within or across problems), the underlying principle (commutativity or operation inversion), or mere exposure to a shortcut is the best method to induce spontaneous shortcut discovery and use. Thirdly, it is debatable if I really consider transfers (Frensch & Haider, 2008). Even if I use the very wide definition of (Ferguson, 1954) that transfer is any effects on ability to perform a specified task resulting from repetition, the term priming might also be suitable. Assuming that all children probably know these strategies, they only need to be taught to apply them. Therefore, it is more likely to find an activation in an associative network (priming).

A further limitation might be the variation in age range within and across the different studies. Based on the current results a more targeted selection of especially relevant age groups would be possible. Many of the studies were driven by exploratory questions and I assessed elementary school children or adults or both together depending on the experiment. This complicates the comparison of the results. The grade/age was selected according to the investigated content. For example, I assessed adults for investigating the spontaneous shortcut usage in graph processing in order to first explore this novel domain under conditions allowing for low intra- and interindividual variability and a high level of understanding of the instructions. Future work might replicate our result with children. If the results with children turn out to be similar, the graphical equivalent of commutativity-based shortcuts could be used to support the development of an integrated concept in arithmetic.

Conclusion

The purpose of this dissertation was to investigate how elementary school children can be supported in applying their knowledge, especially applying appropriate shortcuts in addition problems in a spontaneous and flexible manner. The educational standard for mathematics claims to strive for this goal of the task-appropriate use of flexible solving strategies. Students should be exposed to multiple procedures as early as possible in school. A total of five experiments tested how spontaneous shortcut use could be supported by contextual factors and measured unobtrusively (i.e., using paper-and-pencil approaches, as well as eye tracking). I succeeded in making spontaneous strategy use measurable (and search for shortcut options), using commutativity-based shortcuts as an example. Additionally, I identified beneficial and hindering contextual factors. The results tentatively suggest that future studies might develop learning material that help the acquisition of skills to spot and apply shortcuts spontaneously by offering alternating blocks of problems with different shortcut options based on the same principle. The incorporation of (bar)graphs and estimation needs further research before implementation can start. Also, future studies should explore ways to gauge the positive effects of instruction (fast and targeted activation of procedural and conceptual knowledge), while avoiding drawbacks (i.e., lack of flexibility).

References

Alfieri, L., Brooks, P. J., Aldrich, N. J., & Tenenbaum, H. R. (2011). Does discovery-based instruction enhance learning? *Journal of Educational Psychology*, *103*, 1–18. doi:10.1037/a0021017

Alibali, M. W. (2005). Mechanisms of change in the development of mathematical reasoning. In R. V. Kail (Ed.), *Advances in child development and behavior (Vol 33).* (Vol. 33, pp. 79–123). San Diego, CA US: Elsevier Academic Press.

Anderson, J. R., & Schunn, C. D. (2000). Implications of the ACT-R learning theory: No magic bullets. In *In R. Glaser (Ed), Advances in instructional psychology: Educational design and cognitive science* (pp. 1–33). Publishers.

Baroody, A. J., Feil, Y., & Johnson, A. R. (2007). An alternative reconceptualization of procedural and conceptual knowledge. *Journal for Research in Mathematics Education*, *38*(2), 115–131. doi:10.2307/30034952

Baroody, A. J., & Gannon, K. E. (1984). The development of the commutativity principle and economical addition strategies. *Cognition and Instruction*, *1*(3), 321–339. doi:10.1207/s1532690xci0103_3

Baroody, A. J., Ginsburg, H. P., & Waxman, B. (1983). Children's use of mathematical structure. *Journal for Research in Mathematics Education*, *14*(3), 156–168. doi:10.2307/748379

Barsalou, L. W. (1999). Perceptual symbol systems. *Behavioral and Brain Sciences*, *22*(04), 577–660.

Barsalou, L. W., Kyle Simmons, W., Barbey, A. K., & Wilson, C. D. (2003). Grounding conceptual knowledge in modality-specific systems. *Trends in Cognitive Sciences*, *7*(2), 84–91. doi:10.1016/S1364-6613(02)00029-3

Barsalou, L. W., & Weimer-Hastings, K. (2005). Situating abstract concepts. In D. Pecher & R. Zwann (Eds.), *Grounding cognition : the role of perception and action in memory, language, and thinking* (1. publ.). Cambridge: Cambridge Univ. Press.

Barth, H., La Mont, K., Lipton, J., Dehaene, S., Kanwisher, N., & Spelke, E. (2006). Non-symbolic arithmetic in adults and young children. *Cognition*, *98*(3), 199–222. doi:10.1016/j.cognition.2004.09.011

Barth, H., Mont, K. L., Lipton, J., & Spelke, E. S. (2005). Abstract number and arithmetic in preschool children. *Proceedings of the National Academy of Sciences of the United States of America*, *102*(39), 14116–14121. doi:10.1073/pnas.0505512102

Bilalić, M., McLeod, P., & Gobet, F. (2008). Inflexibility of experts—Reality or myth? Quantifying the Einstellung effect in chess masters. *Cognitive Psychology*, *56*(2), 73–102. doi:10.1016/j.cogpsych.2007.02.001

Bos, W., Wendt, H., Köller, O., & Selter, C. (2012). *TIMSS 2011 Mathematische und naturwissenschaftliche Kompetenzen von Grundschulkindern in Deutschland im internationalen Vergleich*. Münster: Waxmann Verlag.

Bransford, J. D., Barron, B., Pea, R., Meltzoff, A., Kuhl, P., Bell, P., Stevens, R., Schwartz, D., Vye, N., Reeves, B., Roschelle, J., Sabelli, N. (2006). Foundations and opportunities for an interdisciplinary science of learning. In K. R. Sawyer (Ed.), *The Cambridge Handbook of the Learning Sciences* (pp. 19 – 34). New York: Cambridge University Press.

Brunstein, A., & Krems, J. (2006). Transfer von Lernen. In J. Funke & P. A. Frensch (Eds.), *Handbuch der Psychologie* (pp. 267 – 274). Göttingen: Hogrefe Verlag.

Canobi, K. H. (2005). Children's profiles of addition and subtraction understanding. *Journal of Experimental Child Psychology*, *92*, 220–246. doi:10.1016/j.jecp.2005.06.001

Canobi, K. H. (2009). Concept–procedure interactions in children's addition and subtraction. *Journal of Experimental Child Psychology*, *102*, 131–149. doi:10.1016/j.jecp.2008.07.008

Canobi, K. H., Reeve, R. A., & Pattison, P. E. (2002). Young children's understanding of addition concepts. *Educational Psychology*, *22*, 513–532. doi:10.1080/0144341022000023608

Canobi, K. H., Reeve, R. A., & Pattison, P. E. (2003). Patterns of knowledge in children's addition. *Developmental Psychology*, *39*(3), 521–534. doi:10.1037/0012-1649.39.3.521

Cowan, R., & Renton, M. (1996). Do they know what they are doing? Children's use of economical addition strategies and knowledge of commutativity. *Educational Psychology*, *16*, 407–420. doi:10.1080/0144341960160405

Crowley, K., Shrager, J., & Siegler, R. S. (1997). Strategy discovery as a competitive negotiation between metacognitive and associative mechanisms. *Developmental Review*, *17*(4), 462–489. doi:10.1006/drev.1997.0442

Dover, A., & Shore, B. M. (1991). Giftedness and flexibility on a mathematical set-breaking task. *Gifted Child Quarterly*, *35*, 99–105. doi:10.1177/001698629103500209

Dowker, A., Flood, A., Griffiths, H., Harriss, L., & Hook, L. (1996). Estimation strategies of four groups. *Mathematical Cognition*, *2*(2), 113–135. doi:10.1080/135467996387499

ErEl, H., & Meiran, N. (2011). Mindset changes lead to drastic impairments in rule finding. *Cognition*, *119*, 149–165. doi:10.1016/j.cognition.2011.01.002

Ericsson, K. A. (2008). Deliberate practice and acquisition of expert performance: A general overview. *Academic Emergency Medicine*, *15*(11), 988–994. doi:10.1111/j.1553-2712.2008.00227.x

Ericsson, K. A., & Lehmann, A. C. (1996). Expert and exceptional performance: Evidence of maximal adaptation to task constraints. *Annual Review of Psychology*, *47*(1), 273–305. doi:10.1146/annurev.psych.47.1.273

Ferguson, G. A. (1954). On learning and human ability. *Canadian Journal of Psychology/Revue Canadienne de Psychologie*, *8*(2), 95–112. doi:10.1037/h0083598

Fischer, M. H., Dewulf, N., & Hill, R. L. (2005). Designing bar graphs: orientation matters. *Applied Cognitive Psychology*, *19*(7), 953–962. doi:10.1002/acp.1105

Frensch, P. A., & Haider, H. (2008). Transfer and expertise: The search for identical elements. In H. L. Roediger, III (Ed.), Cognitive Psychology of Memory. Vol. [2] of *Learning and Memory: A Comprehensive Reference* (pp. 579–596). Oxford: Academic Press.

Gallistel, C. R., & Gelman, R. (1992). Preverbal and verbal counting and computation. *Cognition*, *44*(1–2), 43–74. doi:10.1016/0010-0277(92)90050-R

Gallistel, C. R., & Gelman, R. (2000). Non-verbal numerical cognition: from reals to integers. *Trends in Cognitive Sciences*, *4*(2), 59–65. doi:10.1016/S1364-6613(99)01424-2

Gaschler, R. (2009). *Information Reduction as item-general strategy change*. Mathematisch-Naturwissenschaftliche Fakultät II. Retrieved from urn:nbn:de:kobv:11-100185071

Gaschler, R., Vaterrodt, B., Frensch, P. A., Eichler, A., & Haider, H. (2013). Spontaneous usage of different shortcuts based on the commutativity principle. *PLoS ONE*, *8*(9), e74972. doi:10.1371/journal.pone.0074972

Geary, D. C., Boykin, A. W., Embretson, S., Reyna, V., Siegler, R., Berch, D. B., & Graban, J. (2008). Report of the task group on learning processes. In *Task Group Reports of the National Mathematics Advisory Panel*. Retrieved from http://www2.ed.gov/about/bdscomm/list/mathpanel/report/learning-processes.pdf

Gentner, D., & Toupin, C. (1986). Systematicity and surface similarity in the development of analogy. *Cognitive Science*, *10*(3), 277–300. doi:10.1016/S0364-0213(86)80019-2

Gilmore, C. K., McCarthy, S. E., & Spelke, E. S. (2007). Symbolic arithmetic knowledge without instruction. *Nature*, *447*(7144), 589–591.

Gilmore, C. K., McCarthy, S. E., & Spelke, E. S. (2010). Non-symbolic arithmetic abilities and achievement in the first year of formal schooling in mathematics. *Cognition*, *115*(3), 394–406. doi:10.1016/j.cognition.2010.02.002

Gilmore, C. K., & Spelke, E. S. (2008). Children’s understanding of the relationship between addition and subtraction. *Cognition*, *107*(3), 932–945. doi:10.1016/j.cognition.2007.12.007

Ginsburg, H. P., Inoue, N., & Seo, K. H. (1999). Young children doing mathematics: Observations of everyday activities. In J. V. Copley (Ed.), *Mathematics in the early years* (pp. 88 – 99). Reston, Va. : Washington, D.C: National Council of Teachers of Mathematics, U.S.

Haider, H., Eichler, A., Hansen, S., Vaterrodt, B., Gaschler, R., & Frensch, P. (2014). How we use what we learn in Math: An integrative account of the development of commutativity. *Frontline Learning Research*, *2*(1). doi:10.14786/flr.v2i1.37

Haider, H., & Frensch, P. A. (1996). The role of information reduction in skill acquisition. *Cognitive Psychology*, *30*(3), 304–337. doi:10.1006/cogp.1996.0009

Haider, H., & Frensch, P. A. (1999). Eye movement during skill acquisition: More evidence for the information-reduction hypothesis. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *25*(1), 172–190. doi:10.1037/0278-7393.25.1.172

Hannula, M. M., & Lehtinen, E. (2005). Spontaneous focusing on numerosity and mathematical skills of young children. *Learning and Instruction*, *15*(3), 237–256. doi:10.1016/j.learninstruc.2005.04.005

Hannula, M. M., Lepola, J., & Lehtinen, E. (2010). Spontaneous focusing on numerosity as a domain-specific predictor of arithmetical skills. *Journal of Experimental Child Psychology*, *107*(4), 394–406. doi:10.1016/j.jecp.2010.06.004

Hansen, S., Haider, H., Eichler, A., Gaschler, R., Godau, C., & Frensch, P. A. (2014). Fostering formal commutativity knowledge with approximate arithmetic. *submitted manuscript*.

Hatano, G. (1988). Social and motivational bases for mathematical understanding. *New Directions for Child and Adolescent Development*, *1988*(41), 55–70. doi:10.1002/cd.23219884105

Hayes, S. C., Brownstein, A. J., Zettle, R. D., Rosenfarb, I., & Korn, Z. (1986). Rule-governed behavior and sensitivity to changing consequences of responding. *Journal of the Experimental Analysis of Behavior*, *45*(3), 256. doi:10.1901/jeab.1986.45-237

Heinze, A., Star, J. R., & Verschaffel, L. (2009). Flexible and adaptive use of strategies and representations in mathematics education. *ZDM*, *41*(5), 535–540. doi:10.1007/s11858-009-0214-4

Hmelo-Silver, C. E., Duncan, R. G., Chinn, C. A. (2007). Scaffolding and achievement in problem-based and inquiry learning: A response to Kirschner, Sweller, and Clark (2006). *Educational Psychologist*, *42*(2), 99–107. doi:10.1080/00461520701263368

Kaminski, J. A., Sloutsky, V. M., & Heckler, A. F. (2008). Learning theory. The advantage of abstract examples in learning math. *Science*, *320*, 454–5. doi:10.1126/science.1154659

Kim, Y. R., Park, M. S., Moore, T. J., & Varma, S. (2013). Multiple levels of metacognition and their elicitation through complex problem-solving tasks. *The Journal of Mathematical Behavior*, *32*(3), 377–396. doi:10.1016/j.jmathb.2013.04.002

Klein, J. S., & Bisanz, J. (2000). Preschoolers doing arithmetic: The concepts are willing but the working memory is weak. *Can J Exp Psychol*, *54*, 105–16.

Koedinger, K. R., & Anderson, J. R. (1990). Abstract planning and perceptual chunks: Elements of expertise in geometry. *Cognitive Science*, *14*(4), 511–550. doi:10.1207/s15516709cog1404_2

Kruschke, J. K. (2001). Toward a unified model of attention in associative learning. *Journal of Mathematical Psychology*, *45*(6), 812–863. doi:10.1006/jmps.2000.1354

Kruschke, J. K. (2003). Attention in learning. *Current Directions in Psychological Science*, *12*(5), 171. doi:10.1111/1467-8721.01254

Lemaire, P., & Siegler, R. S. (1995). Four aspects of strategic change: Contributions to children’s learning of multiplication. *Journal of Experimental Psychology: General*, *124*(1), 83–97. doi:10.1037/0096-3445.124.1.83

Logan, G. D. (1988). Toward an instance theory of automatization. *Psychological Review*, *95*(4), 492–527. doi:10.1037/0033-295X.95.4.492

Logan, G. D. (1992). Shapes of reaction-time distributions and shapes of learning curves: A test of the instance theory of automaticity. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *18*(5), 883–914.

Lorch Jr., R. F., Lorch, E. P., Calderhead, W. J., Dunlap, E. E., Hodell, E. C., & Freer, B. D. (2010). Learning the control of variables strategy in higher and lower achieving classrooms: Contributions of explicit instruction and experimentation. *Journal of Educational Psychology*, *102*(1), 90–101. doi:10.1037/a0017972

Luchins, A. S. (1942). Mechanization in problem solving: The effect of Einstellung. In J. F. Dashiell (Ed.), *Psychological Monographs* (Vols. 1-6, Vol. 54). Illinois: the American Psychological Association.

Mackintosh, N. J. (1975). A theory of attention: Variations in the associability of stimuli with reinforcement. *Psychological Review*, *82*(4), 276–298. doi:10.1037/h0076778

Matthews, P., & Rittle-Johnson, B. (2009). In pursuit of knowledge: Comparing self-explanations, concepts, and procedures as pedagogical tools. *Journal of Experimental Child Psychology*, *104*(1), 1–21. doi:10.1016/j.jecp.2008.08.004

Mayer, R. E. (2004). Should there be a three-strikes rule against pure discovery learning? *American Psychologist*, *59*(1), 14–19. doi:10.1037/0003-066X.59.1.14

McMullen, J. A., Hannula-Sormunen, M. M., & Lehtinen, E. (2013). Young children's recognition of quantitative relations in mathematically unspecified settings. *The Journal of Mathematical Behavior*, *32*(3), 450–460. doi:10.1016/j.jmathb.2013.06.001

Meyer, J., Taieb, M., & Flascher, I. (1997). Correlation estimates as perceptual judgments. *Journal of Experimental Psychology: Applied*, *3*(1), 3–20. doi:10.1037/1076-898X.3.1.3

Ministry of Education, Youth and Sport of the State of Brandenburg, Senate Administration for Education, Youth and Sports Berlin, Senator for Education and Science Bremen, & Ministry of Education, Science and Culture Mecklenburg-Vorpommern (Eds.). (2004). *Rahmenlehrplan Grundschule Mathematik*. Berlin: Wissenschaft und Technik Verlag. Retrieved from http://www.berlin.de/imperia/md/content/sen-bildung/schulorganisation/lehrplaene/gr_ma_1_6.pdf?start&ts=1394618125&file=gr_ma_1_6.pdf

Mullis, I. V. S., Martin, M. O., Foy, P., & Arora, A. (2012). *TIMSS 2011 International results in mathematics*. Chestnut Hill: Bosten Coleage.

Newcombe, N. S., Ambady, N., Eccles, J., Gomez, L., Klahr, D., Linn, M., Miller, K., Mix, K. (2009). Psychology´s role in mathematics and science education. *American Psychologist*, *64*(6), 538–550. doi:10.1037/a0014813

Perkins, D. N., & Salomon, G. (1992). Transfer of learning. In *International Encyclopedia of Education* (2.ed.). Oxford: Pergamon Press.

Prather, R. W., & Alibali, M. W. (2011). Children's acquisition of arithmetic principles: The role of experience. *Journal of Cognition and Development*, *12*, 332–354. doi:10.1080/15248372.2010.542214

Prather, R. W., & Alibali, M. W. (2009). The development of arithmetic principle knowledge: How do we know what learners know? *Developmental Review*, *29*, 221–248. doi:10.1016/j.dr.2009.09.001

Rescorla, R. A., & Wagner, A. R. (1972). A theory of Pavlovian conditioning: Variations in the effectiveness of reinforcement and nonreinforcement. In A. H. Black & W. F. Prokasy (Eds.), *Classical conditioning II: Current research and theory* (pp. 64 – 99). New York: NY: Appleton-Century-Crofts.

Resnick, L. B. (1992). From protoquantities to operators: Building mathematical competence on a foundation of everyday knowledge. In G. Leinhardt, R. Putnam, & R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching*. Hillsdale, N.J.: L. Erlbaum Associates.

Rittle-Johnson, B., & Siegler, R. S. (1999). Learning to spell: Variability, choice, and change in children's strategy use. *Child Development*, *70*(2), 332.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, *93*(2), 346–362. doi:10.1037/0022-0663.93.2.346

Rittle-Johnson, B., Star, J. R., & Durkin, K. (2012). Developing procedural flexibility: Are novices prepared to learn from comparing procedures? *British Journal of Educational Psychology*, *82*(3), 436–455. doi:10.1111/j.2044-8279.2011.02037.x

Robinson, K., & LeFevre, J.-A. (2012). The inverse relation between multiplication and division: Concepts, procedures, and a cognitive framework. *Educational Studies in Mathematics*, *79*, 409–428. doi:10.1007/s10649-011-9330-5

Rosenberg-Lee, M., Barth, M., & Menon, V. (2011). What difference does a year of schooling make?: Maturation of brain response and connectivity between 2nd and 3rd grades during arithmetic problem solving. *NeuroImage*, *57*, 796–808. doi:10.1016/j.neuroimage.2011.05.013

Schneider, M., Rittle-Johnson, B., & Star, J. R. (2011). Relations among conceptual knowledge, procedural knowledge, and procedural flexibility in two samples differing in prior knowledge. *Developmental Psychology*, *47*(6), 1525–1538.

Schwartz, D. L., & Bransford, J. D. (1998). A time for telling. *Cognition and Instruction*, *16*(4), 475–522. doi:10.2307/3233709

Sherman, J., & Bisanz, J. (2009). Equivalence in symbolic and nonsymbolic contexts: Benefits of solving problems with manipulatives. *Journal of Educational Psychology*, *101*(1), 88–100.

Shrager, J., & Siegler, R. S. (1998). SCADS: A model of children's strategy choices and strategy discoveries. *Psychological Science*, *9*(5), 405–410. doi:10.1111/1467-9280.00076

Siegler, R. S. (1996). *Emerging minds: The process of change in children's thinking*. New York: Oxford University Press.

Siegler, R. S., & Jenkins, E. (1989). *How children discover new strategies* (Vol. xiv). Hillsdale, NJ, England: Lawrence Erlbaum Associates, Inc.

Siegler, R. S., & Stern, E. (1998). Conscious and unconscious strategy discoveries: A microgenetic analysis. *Journal of Experimental Psychology: General*, *127*, 377–397. doi:10.1037/0096-3445.127.4.377

Siegler, R. S., & Svetina, M. (2006). What leads children to adopt new strategies? A microgenetic/cross-sectional study of class inclusion. *Child Development*, *77*(4), 997–1015. doi:10.1111/j.1467-8624.2006.00915.x

Singley, M. K., & Anderson, J. R. (1985). The transfer of text-editing skill. *International Journal of Man-Machine Studies*, *22*(4), 403–423. doi:10.1016/S0020-7373(85)80047-X

Star, J. R., & Rittle-Johnson, B. (2008). Flexibility in problem solving: The case of equation solving. *Learning and Instruction*, *18*, 565–579. doi:10.1016/j.learninstruc.2007.09.018

State Institute for School Development Stuttgart, & Ministry of Culture, Youth and Sport of the State of Baden-Württemberg (Eds.). (2004). Bildungsstandards für Mathematik. In *Bildungsplan 2004 Grundschule*. Ditzingen: Philipp Reclam Jun.

Stern, E. (1992). Spontaneous use of conceptual mathematical knowledge in elementary school children. *Contemporary Educational Psychology*, *17*, 266–277. doi:10.1016/0361-476x(92)90065-7

Törneke, N., Luciano, C., & Salas, S. V. (2008). Rule-governed behavior and psychological problems. *International Journal of Psychology and Psychological Therapy*, *8*(2), 141–156.

Verschaffel, L., Luwel, K., Torbeyns, J., & Dooren, W. V. (2009). Conceptualizing, investigating, and enhancing adaptive expertise in elementary mathematics education. *European Journal of Psychology of Education*, *24*(3), 335–359. doi:10.1007/BF03174765

Waldmann, M. (2006). Konzepte und Kategorien. In J. Funke & P. A. Frensch (Eds.), *Handbuch der Psychologie* (pp. 283–293). Göttingen: Hogrefe Verlag.

Wang, J., Conder, J. A., Blitzer, D. N., & Shinkareva, S. V. (2010). Neural representation of abstract and concrete concepts: A meta-analysis of neuroimaging studies. *Human Brain Mapping*, *31*(10), 1459–1468. doi:10.1002/hbm.20950

Wilson-Mendenhall, C. D., Barrett, L. F., Simmons, W. K., & Barsalou, L. W. (2011). Grounding emotion in situated conceptualization. *Neuropsychologia*, *49*(5), 1105–1127. doi:10.1016/j.neuropsychologia.2010.12.032

Wilson-Mendenhall, C. D., Simmons, W. K., Martin, A., & Barsalou, L. W. (2013). Contextual processing of abstract concepts reveals neural representations of nonlinguistic semantic content. *Journal of Cognitive Neuroscience*, *25*(6), 920–935. doi:10.1162/jocn_a_00361

Godau, C., Haider, H., Vaterrodt, B., Frensch, P. A., & Gaschler, R. (submitted). The downside of direct instruction – Instructing one shortcut can hinder spontaneous usage of another arithmetic shortcut based on the same mathematical principle. *Frontline Learning Research, Mai 2014.*

The downside of direct instruction –

Instructing one shortcut can hinder spontaneous usage of another arithmetic shortcut based on the same mathematical principle

Claudia Godau[1/2], Hilde Haider[3], Bianca Vaterrodt[1], Peter A. Frensch[1/2], & Robert Gaschler[2/4]

1 Humboldt-Universität Berlin

2 Cluster of Excellence: Image Knowledge Gestaltung. An Interdisciplinary Laboratory, Berlin

3 Universität Köln

4 Universität Koblenz-Landau

Corresponding author:

Claudia Godau
Humboldt Universität zu Berlin
Mathematisch - Naturwissenschaftliche Fakultät II
Institut für Psychologie
Unter den Linden 6
10099 Berlin, Germany
Phone: +49- 30 2093 9463
Email: claudia.godau@hu-berlin.de

Author Note

This work was supported by Grant FR 1471/12-1 from the Deutsche Forschungsgemeinschaft (DFG). Some of the results were presented at the Conference of Experimental Psychologists 2013 in Vienna, Austria. We thank Christine Arndt, Kate Könnecke and Maria Wirth for help with data collection.

Abstract

Past work suggests that children might be supported in spontaneously discovering and applying an arithmetic shortcut, if they before could discover a different shortcut based on the same mathematical principle. The current work tests whether transfer from shortcut to shortcut could be reached on a more direct route. Specifically we tested whether direct instruction would strengthen the usage of a shortcut based on the commutativity principle and whether it would support children in discovering and using a different shortcut based on the same principle later on. In line with recent work on set effects of instructions in discrimination learning, our results suggest that direct instructions hinder rather than support spontaneous discovery and usage of arithmetic shortcuts.

Introduction

In the field of mathematical education transfer of knowledge is important. Children learn mathematical principles and then need to use this knowledge, when they solve arithmetic problems. In this context we investigated the influence of instruction on the interplay of two different shortcuts, both based on the same principle. As test case we used the commutativity principle. In Germany this principle and the corresponding shortcuts are taught in the first grade, but even toddlers have at least some understanding of the concept of commutativity before entering school (Canobi, Reeve, & Pattison, 2003; Cowan & Renton, 1996; Resnick, 1992; Wilkins, Baroody, & Tiilikainen, 2001).

Rittle-Johnson et al. (2001) proposed an iterative model for the development of procedural and conceptual knowledge in mathematics and especially for commutativity. Haider and colleagues (2014) showed that calculation abilities and conceptual knowledge increase during the elementary school. For example, third to fourth graders had ample prior practice in school in using commutativity-based shortcuts. Gaschler and colleagues (2013) let children from grade 2 to grade 7 and university students solve three-addends addition problems, which are rarely used in class and found that spontaneous usage of two such shortcuts starts to consistently correlate from grade three onwards. Additionally, researchers have shown that in primary school children should associate different strategies based on the same concept and develop the ability to select an efficient strategy for the current problem (e.g., Verschaffel, Luwel, Torbeyns, & Dooren, 2009). Yet it is possible that different forms of procedural knowledge (i.e., shortcut strategies) related to a mathematical principle can be linked (i.e., via common conceptual knowledge) allowing experimenters and teachers to trigger the spontaneous usage of a shortcut by first providing children with the opportunity to discover an easy-to-find shortcut based on the same mathematical principle. Godau and colleagues (2014) tested this assumption and the results suggest that spontaneous shortcut usage triggers knowledge about different shortcuts based on the same principle.

Aiming at transfer between strategies, it should be helpful to ensure that the first presented strategy is recognized and used. In past work, we relied on participants to spontaneously discover a first shortcut, before they were provided the chance to discover and use a second one. A direct activation of the first shortcut might be more efficient to support the second one. Therefore, the goal of the presented experiment was to test whether an explicit instruction about the commutativity based addends-compare strategy also would increase the usage of a different commutativity-based shortcut (i.e., the ten-strategy). If, for instance, a student receives the problem 8 + 5 + 7 = ?, and then 5 + 7 + 8 = ?, he / she can refrain from calculating the second problem presupposed he / she recognizes the applicability of the commutativity principle (i.e. ‘addends-compare strategy’). The commutativity principle enables students to flexibly change the order of addends within a problem. The second commutativity-based shortcut we used is the ‘Ten-strategy’. For instance, given the problem 7 + 4 + 6, it might be easier to calculate (6 + 4) + 7 because 6 + 4 adds up to 10 (i.e. ‘Ten-strategy’). We decided to only test fourth graders. This should maximize the chance to obtain such a transfer effect, because the two commutativity-based shortcuts starts to correlate from grade three onwards (Gaschler et al., 2013). In addition, we also tested adults (university students) in order to check whether the effects would hold beyond the context of primary school.

An emerging consensus is that people learn best through some form of guided discovery, which combines exploration and instruction (e.g. Alfieri, Brooks, Aldrich, & Tenenbaum, 2011). Instruction should not only allow and foster the development of a variety of strategies and fluency in the use of these strategies, but should also stimulate and help children to employ these strategies adaptively. While instructions are effective in the inducing a specific strategy, they maybe be not the best way to foster adaptive expertise. In primary school, instructions often focus on specific shortcuts and give narrow guidelines of what to do (see ErEl & Meiran, 2011; for an example of detrimental effects of instructions on adult discrimination learning). Specifically, we were interested in knowing whether instruction might be well suited to foster flexible usage of knowledge about mathematical principles. Therefore we tested whether instruction to use one shortcut increases the usage of a different shortcut based on the same mathematical principle.

Method

Participants

We tested 83 school children (one school in Berlin) in the classroom and 76 university students (from Humboldt University; details about the two samples are depicted in Table 1) in a seminar room.

Table 1
Sample data.

Grade	Condition	N (females)	Mean age in years (SD)	Seconds for addends-compare / ten-strategy booklets
Grade 4	non-instructed group	40 (17)	9.8 (.34)	120 / 45
	instructed group	43 (16)	9.9 (.50)	
University students	non-instructed group	21 (8)*	23.6 (5.2)	60 / 25
	instructed group	55 (47)	24.2 (5.4)	

* 11 unspecified

Procedure and Materials

Research procedures of this experiment were approved in a peer review process for applying for public funding of the research (German Research Foundation, DFG) and were completed in accordance with approval from the Institutional Review Board of the Department of Psychology at Humboldt-Universität Berlin. We ensured written informed consent of the parents in collaboration with the schools. University students were enrolled at Humboldt-Universität Berlin and received course credit for participation. Either group was provided with advance information concerning the content of the study (calculating mental arithmetic problems) and was informed that participation was voluntary. Participants were also informed that data analysis would not entail charting person-specific results (i.e., names were not collected with the data).

We subsequently presented material with two different commutativity-based shortcut options in order to test for transfer between (a) using a first instructed commutativity-based strategy and (b) applying a second strategy based on the same mathematical principle in later material. Booklets providing the opportunity to use the addends-compare strategy (addends-compare booklets) and booklets allowing for the ten-strategy (ten-strategy booklets) were both accompanied by baseline booklets lacking such shortcut options. As previously mentioned, the addends-compare strategy consists of comparing the addends between successive problems in order to avoid calculation in problems that have the same addends (in different order) as the previously solved problem (e.g., $5 + 8 + 3 = ?$ after $3 + 5 + 8 = ?$). Problems with the opportunity to use the ten-strategy were, for instance, $4 + 3 + 6$ or $7 + 6 + 3$ (for more information about the material see Gaschler et al., 2013). In these problems the first and the last addend add up to 10. It is therefore advisable not to add from left to right, but rather apply commutativity and start with the first and last number.

Three-addend problems were used, because we wanted to establish usage of the commutativity principle with unfamiliar problems. Even though children had received instructions on the commutativity principle during normal schooling, they had been rarely conducted three-addend problems. It is debatable if three-addend problems imply only the commutativity principle or additionally also the associativity principle. Some researchers refer to the associativity principle instead of commutativity if an addition or multiplication problem contains more than two addends or factors (Geary et al., 2008). Other researchers (Canobi, Reeve, & Pattison, 1998) refer to commutativity as the property that problems containing the same terms in a different order have the same solution (independent of the number of addends), whereas associativity is the property that problems in which terms are decomposed, and recombined in different ways, have the same answer [$(a + b) + c = a + (b + c)$]. In line with the definition provided by WIKIPEDIA ("Associative property" 2014) we use the term commutativity for our arithmetic problems as they involve changes in the order of operands in the equation. Associativity refers to the issue that in an expression with two or more subsequent occurrences of the same associative operator, the order in which the operations are performed does not matter as long as the sequence of the operands is not changed. Rearranging the parentheses will not change the value of the expression, e.g. $(5 + 2) + 1 = 5 + (2 + 1) = 8$. Yet, in commutativity, the operands commute – they change in order. Commutativity justifies changing the order or sequence of the operands within an expression while associativity does not. $(5 + 2) + 1 = (2 + 5) + 1$ is referred to as an example of commutativity, but not of associativity, because the operand sequence changed when the 2 and 5 switched places.

We administered the same set of booklets (some of which offered commutativity-based shortcut options) twice in the same order (i.e., first block and second block). Between the first and the second presentation one group of participants received an instruction (see below). Each set consisted of four booklets. The first booklet offered addition problems with the possibility to use the addends-compare strategy. Afterwards students had to calculate comparable problems with no shortcut options (baseline). The third booklet contained the opportunity to use the ten-strategy. The last booklet was again a baseline booklet. For all booklets we used two versions of matched difficulty. By using parallel tests, which were balanced between participants we ensured that participants did repeat structurally equivalent booklets while avoiding exact repetitions of specific arithmetic problems. Between Block 1 and Block 2, students in the *instructed group* were reminded of the commutativity principle in addition (that is, they were explicitly told that problems featuring the same addends as a previously calculated problem would not require calculation). Instructions included an example (i.e., do not calculate 4 + 8 + 9 = ? after having solved 4 + 9 + 8 = ?) in order to make sure that the students could relate the instruction to the material in the task. The *non-instructed group* did not receive any further instructions relating to the commutativity principle.

Time to calculate the problems within the booklets was limited (see Table 1 for exact times). Thus, we expected that students would solve more problems when the booklet contained problems that allowed for shortcuts. All students were told that it was not possible to calculate all given problems, so that they were not frustrated by not finishing the whole booklet. We counted the number of problems solved in the given time. If participants use a shortcut, they should solve more problems in booklets with shortcut option as compared to baseline booklets. The dependent variable was the benefit of the booklet with shortcut option compared to the baseline booklet.

Result

We first present the results of the first block for fourth graders and university students. For the analysis, we subtracted individually for each participant the average amount of solved arithmetic problems in the booklet with commutativity-based shortcut option from the average amount of solved problems in the baseline booklet. In Block 1, we assessed spontaneous usage of the shortcuts for both groups, because the instruction followed only afterwards. We did not find significant differences of shortcut benefits between the two groups (F-values in the appendix). Therefor, we collapsed the data for presentation. In Figure 1, we present the benefit in terms of the number of solved problems in the addends-compare booklet compared to the baseline booklet. The addends-compare strategy was not used spontaneously, but for the ten-strategy the benefit was significant for fourth graders, $t(82) = 4.4$; $p < .001$, as well as for university students, $t(75) = 4.3$; $p < .001$. Apparently, the ten-strategy was detected and used more easily than the addends-compare strategy.

We can rule out that warm-up effects can (alone) account for the lack of a benefit on the addends-compare booklet in Block 1. One might speculate that the first booklet (addends-compare booklet) has a disadvantage over the second booklet (the baseline) as it carries the load of providing a warm-up opportunity (of which the baseline booklet then might profit). However, even in Block 2 (after substantial opportunity for warm-up) fourth grader in the uninstructed group did not used the addends-compare strategy spontaneously, $t(39) = .97$; $p = .34$.

For the second block we first checked if the explicit instruction to use the addends-compare strategy had increased its application (i.e., the direct effect of instruction). After the instruction, the fourth graders significantly benefitted from the addends-compare strategy, $t(42) = 2.8$; $p = .01$. The grey bars in Figure 1 depict this average benefit in number of problems solved for the addends-compare and the ten-strategy booklet relative to the baseline booklet for the second block. On a descriptive level, it can be seen that instructed (dark grey) forth graders seem to have profited more when confronted with problems following the addends-compare-strategy than forth graders without such an instruction (light grey). However, this was only true for the addend-compare problems, not for the ten-strategy problems. For university students, Figure 1 suggests a similar pattern. The benefit for the ten-strategy even seemed to be smaller in the instructed as compared to the non-instructed group.

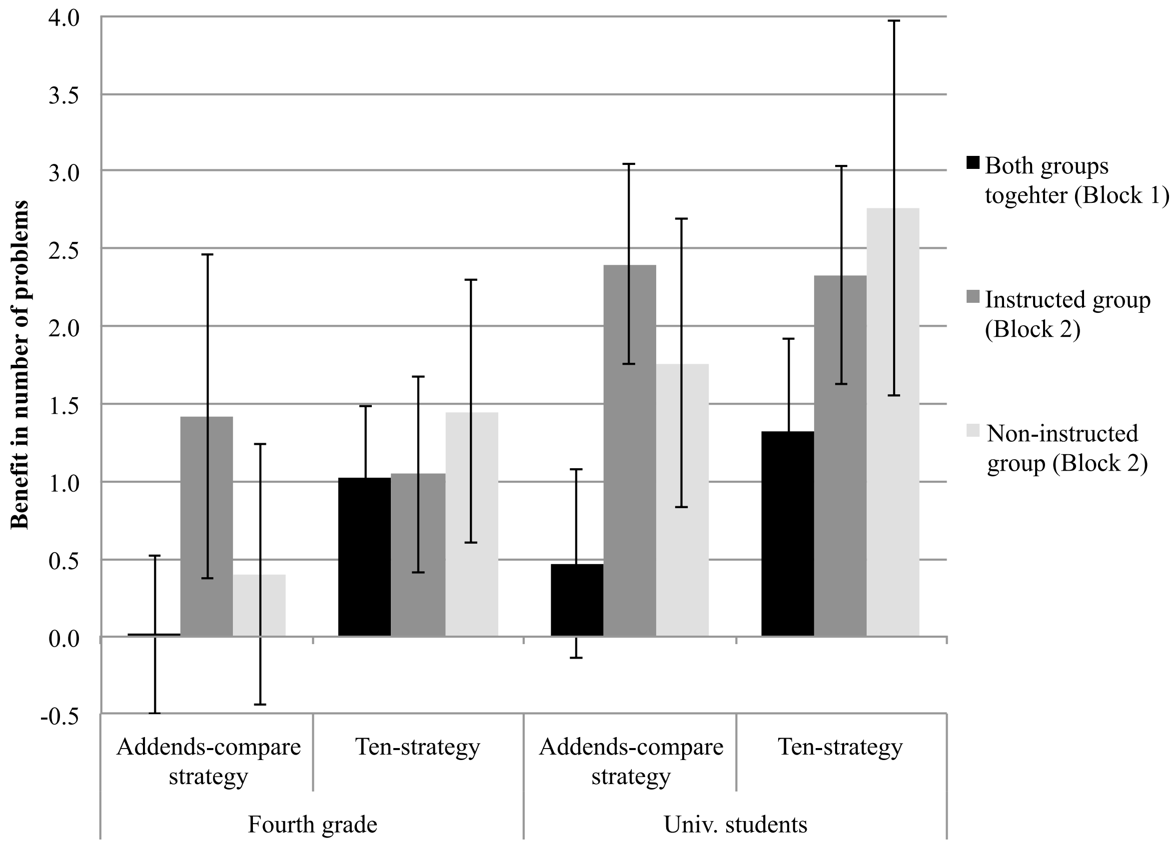


Figure 1. The mean benefit in number of solved arithmetic problems (solved problems with either the addends-compare shortcut or the ten-strategy shortcut minus the number of solved problems in the respective baseline booklets). The error bars refer to the 95% confidence interval of the comparison with zero benefit.

We conducted a 2 (strategy: addends-compare strategy vs. ten-strategy) X 2 (instruction: instructed group vs. non-instructed group) mixed ANOVAs with average benefit in number of solved problem as dependent variable separately for fourth graders and adults. For the fourth graders, the ANOVA revealed no significant main effects, but a significant interaction of strategy and instruction, $F(1, 82) = 4.21$, $p = .05$ $\eta_p^2 = .05$; for all other effects, $F < 1$. This interaction reflected the finding that the instruction to use the addends-compare strategy increased the use of the addends-compare shortcut, while – if at all – exerting a negative effect on ten-strategy usage. For the university students we did not find a significant interaction effect of strategy and instruction ($F(1, 74) = 1.80$, $p = .25$ $\eta_p^2 = .02$) and also the two main effects were not significant (all $F \leq 1$). The pattern of the means did not suggest a positive indirect effect of instruction. The analysis of error rates led to similar findings (see Appendix).

Follow-up analyses suggested that instructions might distort the correlation between strategy benefits measured in the first vs. second block. For the group of fourth graders without the instruction we found a significant (Spearman rank) correlation between the benefit of the ten-strategy from first block to the second one, $r = .41$, $p = .01$. For the group of university students without the instruction, we also found a significant correlation between the benefit of the ten-strategy from first block to the second one, $r = .63$, $p = .01$. Additionally we found for the university students without instruction a significant correlation between the benefit of the addends-compare strategy of Block 1 and the benefit of the ten-strategy in Block 2, $r = .57$, $p = .01$, which supports that the two commutativity-based shortcut are linked. In the conditions with instruction, correlations were absent or less stable (see Table 2).

In summary, we found a correlative relationship between the use of the ten-strategy in the first and the second block except for the fourth graders, who received the instruction. Thus, the instruction differentially influenced the strategies used when solving the problems. In short, the direct instruction had a small, but positive influence on the actual use of addends-compare strategy, but a negative indirect effect on the transfer to the ten-strategy for fourth grader.

Table 2
The Spearman rank correlation between strategy benefits measured in the first vs. second block.

Grade	Condition	Benefit	Addends-compare strategy (Block 1)	Ten-strategy (Block 1)	Addends-compare strategy (Block 2)	Ten-strategy (Block 2)
Grade 4	non-instructed group	addends-compare strategy (Block 1)	1.000	-.105	.259	.200
		ten-strategy (Block 1)		1.000	.078	.406**
		addends-compare strategy (Block 2)			1.000	.254
		ten-strategy (Block 2)				1.000
	instructed group	addends-compare strategy (Block 1)	1.000	.155	.038	.179
		ten-strategy (Block 1)		1.000	-.009	.186
		addends-compare strategy (Block 2)			1.000	.272
		ten-strategy (Block 2)				1.000
University students	non-instructed group	addends-compare strategy (Block 1)	1.000	.414	.041	.565**
		ten-strategy (Block 1)		1.000	.300	.634**
		addends-compare strategy (Block 2)			1.000	.380
		ten-strategy (Block 2)				1.000
	instructed group	addends-compare strategy (Block 1)	1.000	.041	-.037	.044
		ten-strategy (Block 1)		1.000	.289*	.392**
		addends-compare strategy (Block 2)			1.000	.166
		ten-strategy (Block 2)				1.000

* p = .05
** p = .01

Discussion

In this experiment, we tested whether explicitly introducing the addend-compare shortcut would help to spontaneously transfer this shortcut to a different class of problems requiring a different shortcut based on the same mathematical principle. Between the first and the second block we reminded a group of participants of the commutativity-principle and how to apply it on the specific problems ahead.

Direct instruction of one commutativity-based shortcut did not positively affect the spontaneous usage of a different shortcut based on the same mathematical principle. Rather, the opposite seemed to be the case. Instruction of the addends-compare strategy seemed detrimental for flexibly spotting and applying the ten-strategy. Children should develop the skills necessary to flexibly spot and apply shortcut strategies spontaneously. Recent studies have focused on the question that one domain-specific contributor to mathematical development is the Spontaneous Focusing On Numerosity (SFON) (Hannula & Lehtinen, 2005; Hannula, Lepola, & Lehtinen, 2010), investigating how and when children spontaneously recognize and use quantitative relations in everyday situation. The spontaneous focusing on numerosity provides novel insight into children's mathematical thinking and furthers the understanding of how children recognize and utilize mathematical aspects when not explicitly guided to do so (McMullen, Hannula-Sormunen, & Lehtinen, 2013).

Gaschler and colleagues (2013) previously found that the spontaneous (non-instructed) usage of the addends-compare strategy and the ten-strategy is positively correlated in the third grade. Given the current findings, one can only speculate why we did not find any (or a negative) effect of the instruction. Conceivably, the usage of either of the two commutativity-based strategies can in principle foster the usage of the other strategy, but this positive effect is counteracted and cancelled out by negative consequences of instruction. The instruction of only one specific strategy might have diminished the flexibility to spot and apply other shortcut options based on the same mathematical principle. Compliance might have led to a rigid execution of only the instructed procedure rather than have primed the search for other strategies that are conceptually related to the instructed principle. For example, in research about rule-governance, participants who were given the rule as an instruction, consistently showed greater difficulty to adjust to the new conditions than the non-instructed participants (Hayes, Brownstein, Zettle, Rosenfarb, & Korn, 1986; Törneke, Luciano, & Salas, 2008). In line with this view ErEl and Meiran (2011) showed that rule finding (new stimuli, new rule) was impaired by previously applying an instructed (rather than a self-discovered) rule in a discrimination learning procedure.

Surprisingly the results also showed that the addends-compare strategy was not used spontaneously in Block 1, but fourth grader as well as university students spontaneously used the ten-strategy. On the one hand this could already be a transfer effect from one commutative-based shortcut to another one, because we offered the addends-compare strategy booklets always first. On the other hand the ten- strategy could be an easier one. This is more plausible, because in the second block the addends-compare strategy was still not used in the non-instructed group, so a pure trainings-effect could be excluded.

We did not find an indirect instruction effect concerning the transfer. This is in line with work reporting less transfer in a more specific instruction condition than in a more general one (Kaminski, 2008). Focusing the instruction on a particular implementation of an arithmetic principle – which will be practiced on the problems ahead, might lead to a reduction in procedural variability (Siegler, 1996), especially in younger children. The use of multiple strategies is an important developmental milestone (Siegler, 1996) and is associated with greater transfer performance and greater responsiveness to instruction (Alibali & Goldinmeadow, 1993; Siegler, 1996). We suggest that a further milestone lies in balancing between (a) spending time on processing arithmetic problems in a well-established manner vs. (b) on searching for potential shortcut options (Godau et al., submitted).

Taken together, explicitly introducing a certain strategy does not seem to be an efficient way to help students to flexibly spot and apply shortcut options. But instruction and spontaneous flexibility must not be ruled out. One option could consist in a more general instruction to encourage the metacognition. Because a metacognitive system allows for the discovery of new strategies by detecting ways in which existing knowledge can be recombined into new strategies (Robinson & LeFevre, 2012). Future studies could investigate, whether it also supports the application of already learned strategies. It is important for developing metacognitive activities and incorporating them into school curriculum. At the individual level, students are limited in their ability to self-monitor and self-evaluate their problem solving. Kim, Park, Moore and Varma (2013) re-conceptualized metacognition on multiple levels and found that the social level of metacognition is important to potentially overcome individual limitations through feedback and criticism from others.

References

Alfieri, L., Brooks, P. J., Aldrich, N. J., & Tenenbaum, H. R. (2011). Does discovery-based instruction enhance learning? *Journal of Educational Psychology*, *103*, 1–18. doi:10.1037/a0021017

Alibali, M. W., & Goldinmeadow, S. (1993). Gesture-speech mismatch and mechanisms of learning: What the hands reveal about a child′s state of mind. *Cognitive Psychology*, *25*(4), 468–523. doi:10.1006/cogp.1993.1012

Associative property. (2014, June 6). In *Wikipedia, the free encyclopedia*. Retrieved from http://en.wikipedia.org/w/index.php?title=Associative_property&oldid=573574650

Canobi, K. H., Reeve, R. A., & Pattison, P. E. (1998). The role of conceptual understanding in children’s addition problem solving. *Developmental Psychology*, *34*(5), 882–891. doi:10.1037/0012-1649.34.5.882

Canobi, K. H., Reeve, R. A., & Pattison, P. E. (2003). Patterns of knowledge in children’s addition. *Developmental Psychology*, *39*(3), 521–534. doi:10.1037/0012-1649.39.3.521

Cowan, R., & Renton, M. (1996). Do they know what they are doing? Children’s use of economical addition strategies and knowledge of commutativity. *Educational Psychology*, *16*, 407–420. doi:10.1080/0144341960160405

ErEl, H., & Meiran, N. (2011). Mindset changes lead to drastic impairments in rule finding. *Cognition*, *119*, 149–165. doi:10.1016/j.cognition.2011.01.002

Gaschler, R., Vaterrodt, B., Frensch, P. A., Eichler, A., & Haider, H. (2013). Spontaneous usage of different shortcuts based on the commutativity principle. *PLoS ONE*, *8*(9), e74972. doi:10.1371/journal.pone.0074972

Geary, D. C., Boykin, A. W., Embretson, S., Reyna, V., Siegler, R., Berch, D. B., & Graban, J. (2008). Report of the task group on learning processes. In *Task Group Reports of the National Mathematics Advisory Panel*. Retrieved from http://www2.ed.gov/about/bdscomm/list/mathpanel/report/learning-processes.pdf

Godau, C., Haider, H., Hansen, S., Schubert, T., Frensch, P. A., & Gaschler, R. (2014). Spontaneously spotting and applying shortcuts in arithmetic – A primary school perspective on expertise. *Frontiers in Psychology*, *5*, doi:10.3389/fpsyg.2014.00556

Hannula, M. M., & Lehtinen, E. (2005). Spontaneous focusing on numerosity and mathematical skills of young children. *Learning and Instruction*, *15*(3), 237–256. doi:10.1016/j.learninstruc.2005.04.005

Hannula, M. M., Lepola, J., & Lehtinen, E. (2010). Spontaneous focusing on numerosity as a domain-specific predictor of arithmetical skills. *Journal of Experimental Child Psychology*, *107*(4), 394–406. doi:10.1016/j.jecp.2010.06.004

Hayes, S. C., Brownstein, A. J., Zettle, R. D., Rosenfarb, I., & Korn, Z. (1986). Rule-governed behavior and sensitivity to changing consequences of responding. *Journal of the Experimental Analysis of Behavior*, *45*(3), 256. doi:10.1901/jeab.1986.45-237

Kim, Y. R., Park, M. S., Moore, T. J., & Varma, S. (2013). Multiple levels of metacognition and their elicitation through complex problem-solving tasks. *The Journal of Mathematical Behavior*, *32*(3), 377–396. doi:10.1016/j.jmathb.2013.04.002

McMullen, J. A., Hannula-Sormunen, M. M., & Lehtinen, E. (2013). Young children's recognition of quantitative relations in mathematically unspecified settings. *The Journal of Mathematical Behavior*, *32*(3), 450–460. doi:10.1016/j.jmathb.2013.06.001

Resnick, L. B. (1992). From protoquantities to operators: Building mathematical competence on a foundation of everyday knowledge. In G. Leinhardt, R. Putnam, & R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching*. Hillsdale, N.J.: L. Erlbaum Associates.

Robinson, K., & LeFevre, J.-A. (2012). The inverse relation between multiplication and division: Concepts, procedures, and a cognitive framework. *Educational Studies in Mathematics*, *79*, 409–428. doi:10.1007/s10649-011-9330-5

Siegler, R. S. (1996). *Emerging minds: The process of change in children's thinking*. New York: Oxford University Press.

Törneke, N., Luciano, C., & Salas, S. V. (2008). Rule-governed behavior and psychological problems. *International Journal of Psychology and Psychological Therapy*, *8*(2), 141–156.

Verschaffel, L., Luwel, K., Torbeyns, J., & Dooren, W. V. (2009). Conceptualizing, investigating, and enhancing adaptive expertise in elementary mathematics education. *European Journal of Psychology of Education*, *24*(3), 335–359. doi:10.1007/BF03174765

Wilkins, J. L., Baroody, A. J., & Tiilikainen, S. (2001). Kindergartners' understanding of additive commutativity within the context of word problems. *Journal of Experimental Child Psychology*, *79*, 23–36. doi:10.1006/jecp.2000.2580

Appendix

For the additional analyses that the two groups (instructed vs. non-instructed) did not differ in the first block, we present the results of the ANOVA between groups separately for each grade and strategy in Table A1.

Table A1

ANOVA between groups (Block 1 vs. 2) separately for each grade and strategy.

Grade	Benefit of	*F*	*p*
Grade 4	ten-strategy	2.78	.10
	addends-compare strategy	1.60	.21
University students	ten-strategy	1.95	.17
	addends-compare strategy	0.14	.71

For the additional analyses of the error rates, we present the results of the error analysis, focusing on percent error (Table A2). A 2 (problem type: addends-compare/ten-strategy vs. baseline) X 2 (instruction: instructed group vs. non-instructed group) mixed ANOVA for each grade separately showed significant main effect for the ten-strategy versus baseline for the university students in the first block, $F(1, 74) = 9.37$, $p = .01$, $\eta p^2 = .11$, reflecting that more errors occurred in baseline problems as compared to ten-strategy problems and a main effect of instruction for the ten-strategy, $F(1, 74) = 3.92$, $p = .05$, $\eta p^2 = .05$, reflecting that more errors occurred in the non-instructed group as compared to instructed group. In the second block we found a main effect for the addends-compare strategy versus baseline for the fourth graders $F(1, 81) = 5.42$, $p = .05$, $\eta p^2 = .06$, which showed that they made less errors, in booklets, in which the ten-strategy could be used.

Table A2
Error rates per strategy and grade.

		% error											
		Block 1						Block 2					
		Baseline	Addends-compare	*p*	Baseline	Ten-strategy	*p*	Baseline	Addends-compare	*p*	Baseline	Ten-strategy	*p*
fourth grade	instructed group	5.09	6.36	0.36	5.13	4.00	0.43	8.32	7.17	0.45	5.51	4.13	0.35
	non instructed group	4.30	4.73	0.66	4.72	6.10	0.56	6.79	3.59	0.01	4.52	4.92	0.81
University students	instructed group	3.95	3.90	0.95	2.70	1.23	0.12	4.81	3.65	0.25	4.78	1.93	0.01
	non instructed group	2.41	4.62	0.21	6.31	2.21	0.02	6.23	5.15	0.54	4.10	3.26	0.59

Godau, C., Haider, H., Hansen, S., Schubert, T., & Gaschler, R. (2014). Spontaneously spotting and applying shortcuts in arithmetic—a primary school perspective on expertise. *Cognition*, *5*, 556. doi:10.3389/fpsyg.2014.00556

frontiers in
PSYCHOLOGY

ORIGINAL RESEARCH ARTICLE
published: 10 June 2014
doi: 10.3389/fpsyg.2014.00556

Spontaneously spotting and applying shortcuts in arithmetic—a primary school perspective on expertise

Claudia Godau[1,2]*, Hilde Haider[3], Sonja Hansen[3], Torsten Schubert[1,2], Peter A. Frensch[1,2] and Robert Gaschler[2,4]

[1] Department of Psychology, Humboldt-Universität zu Berlin, Berlin, Germany
[2] Cluster of Excellence: Image Knowledge Gestaltung, an Interdisciplinary Laboratory, Berlin, Germany
[3] Department of Psychology, Universität Köln, Köln, Germany
[4] Department of Psychology, Universität Koblenz-Landau, Landau, Germany

Edited by:
Merim Bilalic, Alpen Adria University Klagenfurt, Austria

Reviewed by:
Ann Dowker, University of Oxford, UK
Lieven Verschaffel, University of Leuven, Belgium

****Correspondence:***
Claudia Godau, Institut für Psychologie, Fakultät für Lebenswissenschaften, Humboldt Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany
e-mail: claudia.godau@hu-berlin.de

One crucial feature of expertise is the ability to spontaneously recognize where and when knowledge can be applied to simplify task processing. Mental arithmetic is one domain in which people should start to develop such expert knowledge in primary school by integrating conceptual knowledge about mathematical principles and procedural knowledge about shortcuts. If successful, knowledge integration should lead to transfer between procedurally different shortcuts that are based on the same mathematical principle and therefore likely are both associated to the respective conceptual knowledge. Taking commutativity principle as a model case, we tested this conjecture in two experiments with primary school children. In Experiment 1, we obtained eye tracking data suggesting that students indeed engaged in search processes when confronted with mental arithmetic problems to which a formerly feasible shortcut no longer applied. In Experiment 2, children who were first provided material allowing for one commutativity-based shortcut later profited from material allowing for a different shortcut based on the same principle. This was not the case for a control group, who had first worked on material that allowed for a shortcut not based on commutativity. The results suggest that spontaneous shortcut usage triggers knowledge about different shortcuts based on the same principle. This is in line with the notion of adaptive expertise linking conceptual and procedural knowledge.

Keywords: expertise, numerical cognition, arithmetic, commutativity, spontaneous strategy application

INTRODUCTION

Expertise has various manifestations and could be defined as consistently superior performance within a specific domain relative to novices and relative to other domains (Ericsson and Lehmann, 1996). The development of expertise in real-world domains involves a complex interplay of changes in perception, categorization, memory, problem solving, coordination, skilled action, and other components of human cognition (Palmeri et al., 2004). Expert's flexibility has been frequently discussed and there exist two contradictory perspectives. Research on creativity and skill acquisition has been used to illustrate that more knowledge can make one less flexible (i.e., Luchins, 1942; Logan, 1988). However, research on expertise suggested that experts are more flexible and creative in their thought patterns (see summary in Bilalić et al., 2008a). Both options might be possible depending on the expertise level and the problem difficulty. Investigating chess experts Bilalić et al. (2008a) found that "super experts" were flexible and find the optimal solution first or at least find it quickly after perceiving a salient but non-optimal solution.

Here, we focus in the domain of mathematics on spontaneously spotting and applying shortcuts in arithmetic and whether with further experience students become increasingly able to generate rapid adequate actions with less and less effort (Ericsson, 2008). Mathematic students used significantly larger numbers of appropriate strategies than adults with less expertise (Dowker et al., 1996). Experts have to be able to recognize spontaneously and without instruction that a specific element of their knowledgebase can be applied in a specific situation. It would not suffice if they possessed elaborate conceptual knowledge as well as procedures to apply it, but needed to wait for someone to tell them that the knowledge can be applied in the given situation. This someone would rarely drop by.

In recent years, research in primary school arithmetic has started to tackle this issue for a domain in which everyone should acquire elaborate knowledge. Learning about mathematical principles and procedures should lead to knowledge that can be applied across a wide range of situations (e.g., Hatano and Oura, 2003). Given the role of self-guided learning and performance in the development of mathematical abilities and concepts, recent studies have focused on the question how and when children spontaneously recognize that an everyday situation can be tackled by mathematical thinking (Hannula and Lehtinen, 2005; Hannula et al., 2010; McMullen et al., 2011). Furthermore, children should develop the skills necessary to flexibly spot and apply shortcut strategies spontaneously. It is not sufficient if they can apply a shortcut when explicitly told to do so. Adaptive expertise (Verschaffel et al., 2009) includes to autonomously regulate whether (a) to solve an arithmetic problem in a standard way or to (b) search for / apply a shortcut.

Taking the commutativity principle as a model case, past research has explored how children spontaneously spot and apply shortcuts that allow saving effort in addition problems by flexibly changing the order of addends. Wealth of research has shown that children have at least some understanding of the concept of commutativity before entering school (Baroody and Gannon, 1984; Resnick, 1992; Cowan and Renton, 1996; Wilkins et al., 2001; Canobi et al., 2003). After interviewing elementary school children how they solved problems with two addends, (Baroody et al., 1983) report an extensive use of commutativity. During development children increasingly integrate conceptual knowledge about mathematical principles and procedural knowledge about shortcuts (Haider et al., 2014). Knowledge integration should lead to transfer between procedurally different shortcuts that are based on the same mathematical principle and therefore likely both associated to the respective conceptual knowledge. In a first step, (Gaschler et al., 2013) provided a correlative study to explore this idea. They assessed spontaneous usage of two procedurally different shortcuts that are both based on the commutativity principle in children of different age. While shortcut usage was observed from second grade onwards, correlations between the usage of the two different shortcuts only emerged by grade four. In the current study we aimed at moving beyond correlational data. We tested whether being exposed to one commutativity-based shortcut helps to spot and apply a different shortcut option based on the same mathematical principle. Note that in a parallel line of research, we have observed that instructions do not seem to do the job. Instructing children to use one specific shortcut does hinder rather than assist them in spontaneously spotting and applying a different shortcut based on the same mathematical principle later on (Godau et al., submitted). Instructions about specific procedures might corrupt flexibility in shortcut usage (cf. ErEl and Meiran, 2011). Even when participants knew that a formerly instructed rule would no longer apply, they found it difficult to search for different shortcut options (see also Bilalić et al., 2008a,b; Bilalić and McLeod, 2014). Therefore, in the current work we focused on spontaneous use of the strategies. We explored whether it is possible to foster the discovery and application of shortcut strategies by transfer between different non-instructed shortcut strategies that are based on the same mathematical principle. Note that according to Baroody and Gannon (1984) understanding of commutativity was not evident in all those who invented shortcuts, but in all those who comprehend addition as a binary rather than as a unary operation. The unary view would suggest that one number is added to another, rather than that they are added together.

Specifically, the commutativity principle enables students to flexibly change the order of addends within a problem. For instance, given the problem $4 + 7 + 6$, it might be easier to calculate $(6 + 4) + 7$ ($6 + 4$ adds up to 10 which makes it easy to finally add 7, i.e., "Ten-strategy"). One can also use commutativity across problems. If, for instance, a student receives the problem $8 + 5 + 7 =$?, and then $5 + 7 + 8 =$?, he/she can refrain from calculating the second problem presupposed he / she recognizes the applicability of the commutativity principle (i.e., "addends-compare strategy"). Three-addends problems were used, because we wanted to investigate usage of the commutativity principle with unfamiliar problems. It is debatable if three-addends problems imply only the commutativity principle or additionally also the associativity principle. Associativity is the property that problems in which terms are decomposed, and recombined in different ways, have the same answer $[(a + b) + c = a + (b + c)]$. In the problems we used, children have to change the order of the addends $[a + b + c = (a + c) + b]$, because otherwise it is not possible to add $a + c$ first. Commutativity justifies changing the order or sequence of the operands within an expression while associativity does not.

In Experiment 1, we used eye tracking to explore how children search and apply different commutativity-based shortcuts. Verschaffel et al. (1994) presented third-graders with three-addends problems and assessed eye movements combined with verbal report and found that in 71% of all possible cases commutativity was used. We used a different approach, as we rather were interested in whether children spontaneously start search processes when, after a change in the material one shortcut option is no longer present. The findings suggested that being offered an opportunity to apply one commutativity-based shortcut can help to search for and apply a different shortcut based on the same principle when the first one is no longer feasible. In Experiment 2, we explored whether transfer from shortcut to shortcut might be concept specific: on the one hand, it seems plausible that shortcuts based on the same mathematical principle trigger each other because they are linked to one-another directly or indirectly (as they are both linked to the common conceptual knowledge). This perspective is in line with research suggesting that mathematical knowledge develops in an iterative fashion, with conceptual change influencing procedural change and vice versa (Byrnes and Wasik, 1991; Hiebert and Wearne, 1996; Rittle-Johnson et al., 2001; Waldmann, 2006). For instance, Canobi (2009) showed that children's conceptual advances were predicted by their initial procedural skills. On the other hand, transfer from shortcut to shortcut might occur place for motivational reasons unrelated to the specific shortcut and underlying mathematical principle. After having experienced that task processing can be simplified by a shortcut, one might be more apt to search for and apply *any* shortcut, as one has learned that attractive shortcut options do seem to exist in the material provided.

EXPERIMENT 1

In Experiment 1, we used eye tracking in order to explore the fixation patterns reflecting the usage of shortcut strategies. We were furthermore interested in how fixation patterns reflect how people accommodate to being presented with new sets of arithmetic problems within which the previously feasible shortcut no longer applies (but instead a different shortcut). To this end, children at first had to solve problems that could be facilitated by the ten-strategy (of three addends, the first and the last add up to 10). After that, they were presented with problems that allowed for the use of the addends-compare strategy (some problems contained the same addends as their precursor in different order). Both strategies are based on the commutativity principle.

METHOD EXPERIMENT 1

Participants

Twenty children participated in Experiment 1 (mean age 8.6 years). They were tested individually in a laboratory at Humboldt-Universität, Berlin.

Procedure and Materials

Research procedures of these experiments were approved in a peer review process for applying for public funding (German Research Foundation, DFG) and were completed with approval of the Institutional Review Board of the Department of Psychology at Humboldt-Universität, Berlin. Students were informed about the content of the study and that data analysis would preserve anonymity. We ensured written informed consent of the parents. Children were than tested individually with a 250 Hz video-based eye tracker (SMI RED 250). Packages of six problems in black on a gray background were shown on a 22 TFT monitor, with the student sitting at approximately 50 cm distance. Digits were approximately 0.5 cm wide and 1 cm tall.

Children started with a five-point calibration. Afterwards the experimenter showed a single example problem and explained that the children should utter the result as quickly and as accurately as possible. Children started the main part by working on two screens with six ten-strategy problems each (first and last addend add up to 10). They then completed two screens with addends-compare problems intermixed with baseline problems. Two of six problems per screen contained identical addends in different order as the preceding problem (problems listed in the Supplementary materials). Each problem was presented in one line and consisted of three different addends between 2 and 9 (maximum result was 24; 0 and 1 were excluded as addends). We balanced problem size between the addends-compare problems and the baseline problems so that they were equally difficult for children unless they used the shortcut (for more details Gaschler et al., 2013; Haider et al., 2014).

Children were presented the first screen (of two) with six ten-strategy problems. The experimenter moved the cursor to the right of the equal sign of the first problem and waited for an answer. The answer was immediately entered as the time log of the first key press served to determine the calculation time as the span from the cursor allocation to the first (i.e., two-digit results) key press of entering the result for the current problem. After entering the answer, the experimenter moved the cursor to the next problem. The entered results remained visible on the screen while working on the remaining of the six problems of the package. This was especially important for the work on the two screens with addends-compare problems later on. If they had spotted that the addends of a problem were the commuted version of the preceding problem, that way they were provided with the opportunity to access the solution they had given on the previous problem.

RESULTS

The computerized assessment allowed to track solution times on the level of single problems. As previously mentioned, students calculated 12 ten-strategy problems (Screen 1 and 2) and afterwards worked on yet another 12 problems, four of them allowed for the addends-compare strategy (Screen 3 and 4). **Figure 1** shows the mean solution times per problem for each screen. Students were faster on addends-compare problems as compared to baseline problems. A 2 (screen: first vs. second) × 2 (problem type: addends-compare problem vs. baseline problem) ANOVA with solution times as dependent variable revealed a significant main effect of problem type, [$F_{(1,\ 19)} = 7.46$, $p = 0.01$, $\eta_p^2 = 0.28$]. Neither the main effect of screen, [$F_{(1,\ 19)} = 1.67$, $p = 0.21$, $\eta_p^2 = 0.08$], nor the interaction effect were significant, [$F_{(1,\ 19)} = 0.72$, $p = 0.41$, $\eta_p^2 = 0.04$]. We did not find significant effects when repeating the above analyses with error rate as dependent variable (see Supplementary materials).

The analysis of the eye tracking data suggests that the ten-strategy and the addends-compare strategy can be identified by specific fixation patterns. Using the ten-strategy, adding the first and last addend first to receive the result ten, should be fast and necessitates little fixation time on the outer numbers. Adding the middle number afterwards and uttering the result might therefore result in more fixation time on the middle number relative to the other numbers. **Figure 2** suggests that the percent fixations falling on the middle vs. outer numbers of the three-addends problems are distributed in line with this reasoning. The percentage of fixations on the middle number increased from the first to the second screen of the ten-strategy problems, as students presumably discovered the structure of the problems.

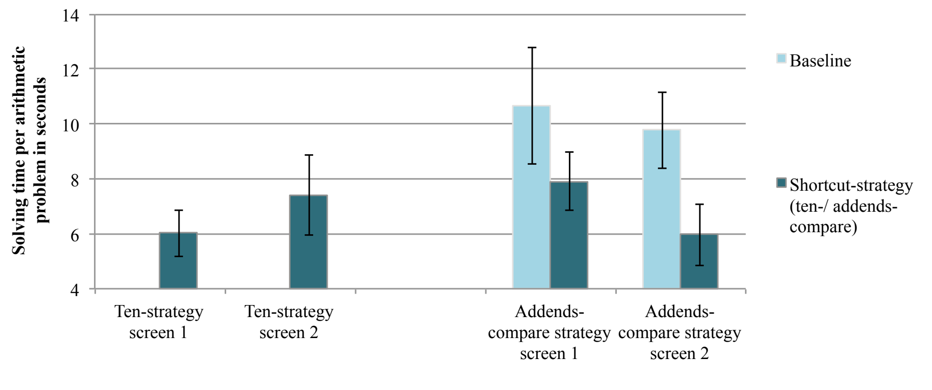


FIGURE 1 | Mean calculation time per arithmetic problem in Experiment 1. Error bars indicate the standard error of the mean.

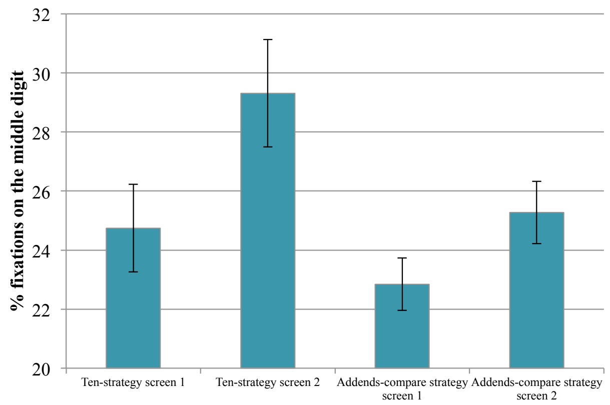


FIGURE 2 | Percent fixation frequency on second addend for ten-strategy and addends-compare strategy booklets. The error bar displays the standard error of the mean.

When the ten-strategy could no longer be used (first screen with addends-compare problems), the percent fixations on the middle digit were low again. Surprisingly, it increased on the second screen with addends-compare problems. A 2 (screen: first vs. second) × 2 (ten-strategy problems vs. addends-compare problems) ANOVA with percentage of fixations falling on the middle number as dependent variable revealed a significant main effect for strategy [$F_{(1, 17)} = 6.02$, $p < 0.05$, $\eta_p^2 = 0.26$]. Children fixated the middle digit more in problems, in which the ten-strategy could be used compared to problems on the addends-compare screens. There was also a significant main effect of screen, [$F_{(1, 17)} = 7.91$, $p = 0.01$, $\eta_p^2 = 0.32$], but no interaction, $F < 1$.

In the ten-strategy problems, addends should be checked within a line in order to identify shortcut options. In contrast, for the addends-compare strategy, it is necessary to compare the addends between the lines. Children should thus not only fixate the addition problem they are currently solving but also the previous one or the subsequent one in order to check whether a set of addends repeats. **Figure 3** presents the mean differences between (a) line fixated and (b) line of current problem. If, for instance, a student during solving a problem was fixating back on the problem in the line before, this would lead to a value of −1 for this particular fixation. While the majority of fixations were on the line of the current problem, some fixations were directed at previous (negative difference) or subsequent (positive difference) problems. We focused on comparing the above index of fixation position between the addends-compare problems and their preceding problems. Thus, addends are identical and only differ in order. We found a significant difference in the index of fixation position for these problems. In line with our assumption, students were fixating ahead on problems preceding the addends-compare problems and fixating back, once a set with identical addends was discovered, [$t_{(18)} = 5.44$, $p < 0.001$].

In addition to identifying eye movement patterns that are specific for the shortcuts we found a significant correlation between the increase of the fixation on the middle digit in the ten-strategy problems (Screen 2—Screen 1) and the time benefit on addends-compare strategy problems $r = 0.49$, $p = 0.05$. Thus, increased usage of the commutativity-based shortcut offered on Screen 1 and Screen 2 might help in spotting and applying the other commutativity-based shortcut offered on Screen 3 and 4.

DISCUSSION

Providing children with the opportunity to spontaneously (without instruction or other hints) use one commutativity-based shortcut might help them to spot and apply another shortcut based on the same mathematical principle once the first one does no longer apply. Furthermore, the eye tracking data are in line with the interpretation that search processes might start once one shortcut no longer applies. We found that children in some cases checked addends of subsequent addition problems in advance (i.e., before uttering the result to the current problem and the allocation of the cursor to the next problem). Note that this implies that the accuracy to attribute calculation time to specific arithmetic problems might be limited in setups in which multiple problems are simultaneously presented. Such arrangements resemble work on arithmetic problems on worksheets in the schooling context. Eye tracking or reliance on aggregate measures from paper-and-pencil versions might both be useful approaches to this variant of the dilemma of external vs. internal validity.

Experiment 1 provided a first hint in line with the idea that there might be transfer from one shortcut to another one. This suggests two different explanations. On the one hand, spontaneously spotting and applying shortcuts on Screen 1 and 2 might affect processing of Screens 3 and 4 on a motivational route. Participants learn that shortcut options seem to exist and can be exploited. This would suggest that such transfer could take place from any easily identifiable shortcut to a second one. On the other hand, transfer might involve specific mathematical knowledge. It might first and foremost take place between shortcuts based on the same mathematical principle. We tried to disentangle these two perspectives in Experiment 2.

EXPERIMENT 2

This experiment focused on the question if the ten-strategy facilitated the usage of the addends-compare shortcut. For this purpose, we compared three conditions: students in the ten-strategy warm-up condition started with the ten-strategy problems followed by problems that allowed for applying the addends-compare strategy (similar to Experiment 1). In the baseline warm-up condition, children worked on material with no shortcut option at all before being transferred to the addends-compare booklet. The inversion warm-up condition started with inversion problems (e.g., $9 + 2 - 2$). Thus, a shortcut *not* based on the commutativity principle was offered first. This was important in order to test whether all shortcut strategies would alter the usage of the addends-compare shortcut simply by motivation children to look for shortcuts. Alternatively, it might be that only the ten-strategy increases the probability to spot the addends-compare strategy, as it is the only shortcut strategy, which is also based on the commutativity principle. It is conceivable that offering problems with an easy-to-find shortcut option (inversion or ten-strategy) might lead students to assume that it is worthwhile to search for shortcut options in general in later material. This could accordingly lead to transfer which is simply based on

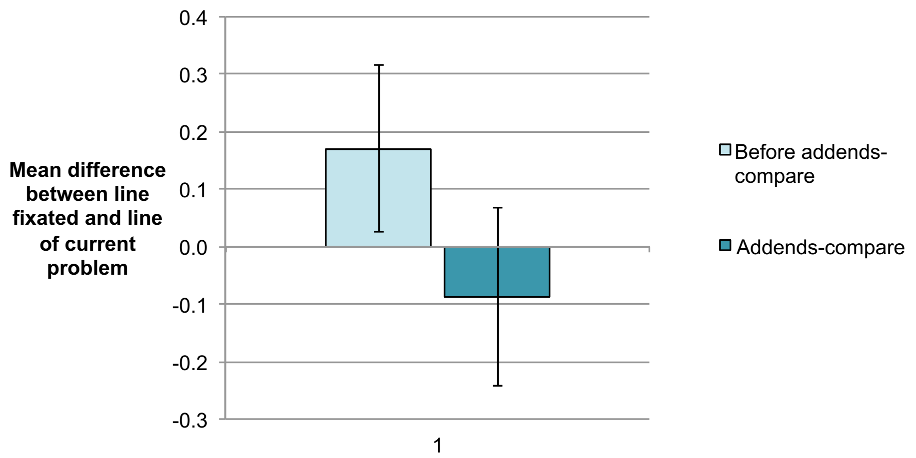


FIGURE 3 | Mean difference between current line fixated and line of current problem. Negative values indicate fixations on preceding problems while positive values result from fixations on subsequent problems. Error bars indicate the 95% CI of the comparison of addends-compare problems vs. preceding problems.

the motivation to search for shortcuts. In contrast, a finding of transfer for the ten-strategy problems but not for the inversion problems would suggest that indeed triggering the basic principle of commutativity is important for transfer to occur.

METHOD EXPERIMENT 2

Participants

We tested 153 children at the end of second grade (most of them were taught in combined classes of first and second grade) and 140 children in third grade. We ensured written informed consent of the parents in collaboration with the schools. Either group was provided with advance information concerning the content of the study (calculating mental arithmetic problems) and was informed that participation was voluntary. Parents and students were also informed that data analysis would preserve anonymity. Data were acquired in a classroom setting with paper and pencil. Gender was balanced as much as possible. Eleven children (second grade) and 20 children from the third grade were excluded by median $\pm$ 3 *MAD*s. The MAD is a robust method to detect outliers by using absolute deviation from the median; for further information see (Leys et al., 2013). For the descriptive data of the sample see **Table 1**.

Procedure and Materials

The arithmetic problems were the same as in Experiment 1 and are listed in the Supplementary materials. Each problem was presented in one line and consisted of three different addends between 2 and 9 (maximum result was 24; 0 and 1 were excluded as addends). The different types of problems were presented as a paper pencil test in separate booklets. As dependent variable we measured the number of problems solved in the booklet that allowed vs. the booklet that did not allow for the addends-compare strategy. We took care that the amount of time provided per booklet was not sufficient to solve all problems so that we could use number of problems solved per time as a dependent variable (see **Table 1** for time provided per booklet).

Table 1 | Sample data and time provided per booklet in Experiment 2.

Grade	Condition/ warm-up	Outliers	*N* (female)	Mean age (*SD*)	Seconds for addends-compare booklets
2	Ten-strategy	4	48 (25)	7.1 (0.69)	240
	Baseline	1	49 (26)	7.1 (0.72)	
	Inversion	6	45 (25)	7.1 (0.62)	
3	Ten-strategy	5	41 (24)	8.0 (0.35)	180*
	Baseline	7	40 (20)	8.2 (0.71)	
	Inversion	8	39 (25)	7.8 (0.64)	

**We started with 210 s and than reduced it after testing one group of students in order to avoid ceiling effects.*

Experimental conditions differed in the warm-up booklet. The ten-strategy warm-up started with problems in which children could use the ten-strategy. The baseline warm-up conditions started with addition problems of comparable size, but that did not include any option for applying the commutativity principle to solve the problems (e.g., $4 + 3 + 5$ or $7 + 6 + 2$). A second control condition, the inversion warm-up condition, started with problems that allowed for a shortcut, but, importantly, not for a commutativity-based one. Inversion problems (e.g., $9 + 2 - 2$) allow refraining from calculation by comparing the numbers involved in the problem mixing addition and subtraction. Thus, while the ten-strategy and addends-compare strategy are both based on the same arithmetic principle, inversion and addends-compare are not. However, on the surface the latter two shortcuts are similar as they both enables students to avoid calculation altogether (in contrast, the ten-strategy does reduce instead of avoid calculation demands).

After the warm-up phase, all children worked on five more booklets. Starting with (1) a booklet, where the addends-compare strategy could be used, they then were presented (2) a baseline booklet with no shortcut opportunities, followed by (3) another booklet, where the addends-compares strategy could be used.

This second addends-compare booklet was applied as we had obtained high variability across students as well as large general practice effects in the first booklets in earlier work (Gaschler et al., 2013). Booklets 4 and 5 served the purpose to control whether the induced shortcut is known and would be used (see **Table 2**). The children in the ten-strategy warm-up condition received another booklet with addition problems allowing for the ten-strategy (4) plus afterwards a matched baseline booklet (5). This was also the case for children of the control condition with the baseline warm-up. The children of the inversion warm-up condition worked for the second time on a booklet with inversion problems (4) followed by a matched baseline booklet (5).

Students were instructed to solve the problems as quickly and as correctly as possible. The time for each booklet was fixed and we counted the number of problems solved and errors. Students were additionally informed that it would be almost impossible to solve all problems during the period of time given for each booklet. As dependent measure we calculated the average time per problem on addends-compare booklets as compared to baseline booklets.

RESULTS

After the short warm-up phase, children were still rather slow in calculating the first set of addends-compare booklets and between students variability was rather high (see **Table 3**). On closer examination, we found that the practice effects were stronger than the effect of problem type. For further analysis we focused on the second addends-compare booklet. We first analyzed the effects of our different warm-up phases on the addends-compare problems. For calculating the addends-compare benefit in second graders, we subtracted for each child the average solution time per problem in Booklet 3 (addends-compare strategy) from the average time per problem in Booklet 2 (baseline). The benefits are depicted in **Figure 4** separately for each of the three conditions in second and third graders. In addition, **Table 3** presents the average time per problem for every booklet for the second and third grade.

For the second graders with the ten-strategy warm-up phase, we observed a significant benefit on the addends-compare strategy problems compared to baseline problems $t_{(47)} = 2.48$, $p = 0.05$. Second graders with the warm-up problems not allowing for any shortcut did not benefit from the addends-compare booklets relative to the baseline booklets. The inversion problems group also did not show such a benefit either. Third graders, however, seemed to use the addends-compare strategy in every warm-up condition. Each of the three warm-up groups significantly benefitted from the addends-compare strategy [ten-strategy: $t_{(40)} = 2.64$, $p = 0.05$; baseline: $t_{(39)} = 3.71$, $p = 0.001$; inversion: $t_{(38)} = 3.79$, $p = 0.001$]. The time used to solve the addends-compare strategy problems was shorter than that needed to calculate the baseline problems.

We calculated a 2 (problem type: baseline vs. addends-compare booklet) × 3 (warm-up condition: ten-strategy vs. baseline vs. inversion warm-up) × 2 (grade: second vs. third grade) mixed ANOVA with mean benefit time as dependent variable. This ANOVA yield significant main effects of problem type [$F_{(1,\ 256)} = 14.98$, $p < 0.001$, $\eta_p^2 = 0.055$] and grade [$F_{(1,\ 256)} = 38.44$, $p < 0.001$, $\eta_p^2 = 0.131$] and a significant three-way interaction of problem type × warm-up condition × grade [$F_{(2,\ 256)} = 3.75$, $p = 0.05$, $\eta_p^2 = 0.028$]. We found neither a significant main effect for warm-up condition, nor other interaction effects (see **Table 4**). The three-way interaction suggests that the different warm-up phases differentially affected second and third graders. Whereas the ten-strategy warm-up increased the probability of applying the addends-compare strategy in second graders, it did not in third graders. The results suggest that shortcut to shortcut transfer specific to the underlying mathematical principle was observed in second graders. Third graders, on the other hand, maybe spontaneously used the addends-compare shortcut anyways and thus did not profit from a prior task with a conceptually related shortcut.

One could argue that second graders did not show transfer from an inversion warm-up to addends-compare problems, because they did not discover the shortcut option in the

Table 2 | The order of the booklets in Experiment 2.

Condition/ Warm-up	Booklet 1	Booklet 2	Booklet 3	Booklet 4	Booklet 5
Ten-strategy	Addends-compare strategy	Baseline	Addends-compare strategy	Ten-strategy	Baseline
Baseline					
Inversion				Inversion	

Table 3 | Mean time per problem and standard deviation analyzed for booklet type and grade in Experiment 2.

Grade	Condition	Warm-up	Booklet 1: Addends-compare strategy (1)	Booklet 2: Baseline	Booklet 3: Addends-compare strategy (2)	Benefit (baseline—addends-compare strategy (2))	Booklet 4: Same as warm-up*	Booklet 5: Baseline (2)
2	Ten-strategy	26.4 (26.8)	28.1 (34.6)	25.2 (23.0)	20.1 (12.0)	5.1 (14.3)	21.9 (22.7)	18.4 (9.2)
	Baseline	23.8 (15.2)	28.3 (24.3)	22.9 (13.4)	22.6 (14.6)	0.3 (5.7)	23.0 (18.4)	22.0 (18.3)
	Inversion	26.3 (29.0)	28.2 (20.4)	25.0 (19.6)	24.4 (16.6)	0.6 (10.7)	17.8 (24.9)	24.4 (21.7)
3	Ten-strategy	10.4 (3.9)	13.2 (3.3)	13.5 (3.7)	12.4 (3.6)	1.1 (2.7)	11.1 (4.6)	12.3 (5.6)
	Baseline	13.9 (7.4)	14.6 (4.5)	15.8 (5.8)	13.3 (2.9)	2.4 (4.2)	13.2 (7.3)	13.4 (5.5)
	Inversion	12.3 (10.3)	15.6 (8.1)	15.8 (6.3)	13.4 (4.5)	2.4 (3.9)	5.9 (5.7)	13.1 (8.2)

*See **Table 2**.

inversion problems. Our manipulation checks do not support this alternative explanation. We analyzed the Booklets 4 and 5 (induction shortcut—and respective baseline). The results suggested that students were capable of using the inversion strategy (see **Table 5**). For the second graders, a 2 (Booklet 4 vs. 5) × 3 (warm-up condition) ANOVA revealed a significant interaction effect of both factors, [$F_{(2,\ 139)} = 3.20$, $p = 0.05$, $\eta_p^2 = 0.044$]. It depended on the warm-up condition, whether the shortcut in Booklet 4 was used.

For the third graders we also found an interaction effect of Booklet 4 vs. 5 and warm-up condition, [$F_{(2,\ 117)} = 15.41$; $p < 0.001$, $\eta_p^2 = 0.208$]. While there was a pronounced inversion effect, surprisingly, neither baseline warm-up condition nor the ten-strategy warm-up condition showed a ten-strategy effect in the booklets administered at the end of the experiment. We did not find relevant effects when repeating the above analyses with error rate as dependent variable, but needless to say we found different error rates in grade two and three (see Supplementary materials).

DISCUSSION

In Experiment 2, we tested whether it is possible to make students to spot and apply a shortcut strategy by first providing an easy-to-find shortcut strategy based on the same mathematical principle vs. one based on a different principle. Our findings suggest that in second graders, transfer was related to the mathematical principle rather than to general motivational factors. There was no indication that second graders were motivated to search for and apply *any* shortcuts after being offered the first one. If the additional conceptual link between the two different strategies is the reason for the transfer, this would support understanding of adaptive expertise as the ability to apply meaningfully learned procedures flexibly and creatively (Hatano and Oura, 2003). The inversion warm-up phase—an easy-to-find shortcut that is not based on commutativity—did not lead to increased usage of the addends-compare strategy. While inversion did not promote transfer, our manipulation check suggested that inversion was indeed used. This is in line with Robinson and Dubé (2009) who found that the inversion shortcut is easier to apply than associativity (which is similar to commutativity). In both studies (Robinson and Dubé, 2009; Dubé and Robinson, 2010), inversion shortcut use was far more frequent than the associativity-based strategy. Focusing on commutativity as model case a limitation of the experiment is that we so far only used one shortcut not based on commutativity (i.e., inversion) in order to differentiate between transfer effects based on motivation vs. on mathematical principles shared by subsequently offered shortcut options. For instance, it would be interesting to know whether the current setup can be turned around with inversion usage as dependent variable and commutativity vs. inversion warm-up as independent variable (cf. Dowker, 2014). Generalizability beyond the specific pairing of shortcuts tested here might for instance depend upon the relative difficulty of shortcuts used as warm-up and dependent variable.

While the results suggest that second graders profited from shortcut-to-shortcut transfer based on commutativity, third graders did not seem to benefit from such extra scaffolding. Spontaneous usage of the addends-compare strategy was not improved further by a warm-up condition with a shortcut-option based on the same mathematical principle. We assume that in this age group, the concept of commutativity is more developed so that extra support is less needed. With further experience, students become increasingly able to rapidly generate adequate actions with less and less effort (Ericsson, 2008). In line with these findings, differences between second and third graders in their mathematical abilities are mirrored in functional changes of the brain. Rosenberg-Lee et al. (2011) examined the behavioral

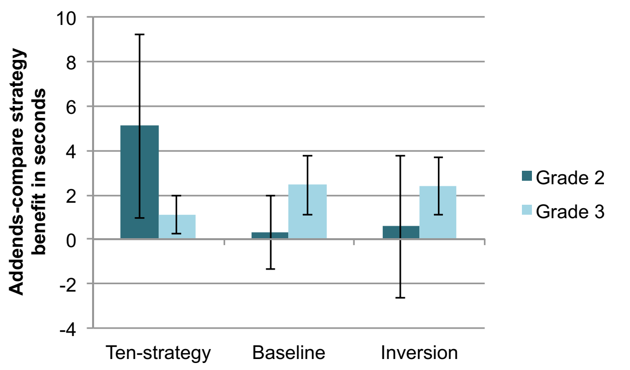


FIGURE 4 | The mean benefit in seconds of booklets allowing for the addends-compare strategy compared to baseline booklets for the three different warm-up conditions (ten-strategy, baseline and inversion) for the second grade (dark gray) and the third grade (light gray) in Experiment 2. The error bar displays the 95% confidence interval of the comparison with zero benefit.

Table 4 | Experiment 2: Results of the ANOVA problem type × grade × condition.

		F	p	η_p^2
Main effect:	Problem type (addends-compare strategy vs. baseline)	14.98	0.00	0.06
	Grade	38.44	0.00	0.13
	Warm-up condition	0.49	0.61	0.00
Inter action:	Problem type (addends-compare strategy) × grade	0.00	0.96	0.00
	Problem type (addends-compare strategy) × warm-up condition	1.57	0.21	0.01
	Warm-up condition × grade	0.14	0.87	0.00
	Problem type (addends-compare strategy) × warm-up condition × grade	3.75	0.02	0.03

Table 5 | Results of the ANOVA problem type × condition separately for grade 2 and 3.

		Grade 2			Grade 3		
		F	p	η_p^2	F	p	η_p^2
Main effect:	Problem type (addends-compare strategy)	4.92	0.03	0.03	35.04	0.00	0.23
	Warm-up condition	0.23	0.80	0.00	1.95	0.15	0.03
Inter action:	Problem type (addends-compare strategy) × warm-up condition	2.97	0.06	0.04	1.73	0.18	0.03

and neurodevelopmental changes between grades 2 and 3 and found that arithmetic complexity was associated with regions implicated in domain-general cognitive control but also regions for numerical arithmetic processing. The results showed that brain response and connectivity relating to an arithmetic task significantly change within the narrow 1-year interval.

GENERAL DISCUSSION

We presume that one crucial feature of expertise is the ability to spontaneously recognize where and when knowledge can be applied to simplify task processing. In some domains, it is necessary for everyday life to develop this ability. Research of expertise showed that experts are more flexible and creative in their thought pattern. For instance, "super experts" were more flexible to find an optimal solution despite distraction by a non-optimal but salient solution of a chess problem (Bilalić et al., 2008a). Players at lower levels of expertise reported that they were looking for a better solution, but their eye movements showed that they continued to look at features related to the solution they had already thought of (Bilalić et al., 2008b). For expertise in object recognition, Harel et al. (2013) developed an interactive framework, which posits that expertise emerges from multiple interactions within and between the visual system and other cognitive systems, such as top-down attention and conceptual memory. The interplay between these other, multiple cognitive processes and perception are often not consciously accessible for the experts themselves (Palmeri et al., 2004).

In some parts of arithmetic, procedural and conceptual knowledge start to develop even before primary school. In the first years of primary school, integration of different fragments of procedural and conceptual knowledge should lead to a knowledge base that allows to spontaneously spot and apply shortcut options already in primary school. If successful, knowledge integration should lead to transfer between procedurally different shortcuts that are based on the same mathematical principle and therefore likely are both associated to the respective conceptual knowledge. For the case of commutativity, we tested whether different strategies that are based on the same principle trigger each other via the concept and so could support flexibility in strategy use. According to the adaptive expertise metaphor (e.g., Hatano, 1988; Star and Rittle-Johnson, 2008; Verschaffel et al., 2009) children first of all need to spontaneously recognize where knowledge can be applied.

Experiment 1 provided first evidence that children who are provided an opportunity to spontaneously spot and apply one shortcut might be more inclined to search for and use a second shortcut, once the first one no longer applies. This is in line with the suggestion to differentiate between (a) quick and accurate routine-based solving from (b) an adaptive use of solution strategies, which draws upon conceptual understanding (Hatano, 1988). Experiment 2 verified that transfer occurred from one shortcut to another. It furthermore specified that this transfer effect was not only based on motivation. While we obtained transfer (at least in second graders) from one commutativity-based shortcut to another commutativity-based shortcut, no transfer was observed between inversion and commutativity. Thus, our results are in line with the view that links between different elements of procedural knowledge and potentially conceptual knowledge (compare Haider et al., 2014) are used to spontaneously spot and apply shortcut options.

Several studies on commutativity have shown that children have at least some understanding of the concept of commutativity before entering school (Siegler, 1989; Resnick, 1992; Cowan and Renton, 1996; Wilkins et al., 2001; Canobi et al., 2003) and already first graders seem to understand the commutativity principle (Canobi et al., 2002). We thus focused on triggering the usage of knowledge rather than knowledge acquisition as such. In primary school, children should link different strategies based on the same concept and develop the ability to select an efficient strategy for the current problem (Verschaffel et al., 2009). As implied by these authors in the adaptive expertise metaphor, the learner should be able to spot and apply options for a shortcut independently without having to rely on instruction or explicit cues. In a similar vein, research on skill acquisition and expertise stresses the importance of linking perceptual skills and principle-knowledge in order to be able to spontaneously spot and apply shortcuts (e.g., Gentner and Toupin, 1986; Koedinger and Anderson, 1990; Haider and Frensch, 1996; Anderson and Schunn, 2000; Bilalić et al., 2008a; Frensch and Haider, 2008). Adaptive strategy use can be regarded as the ability to select procedures that can simplify the solution of a problem (Selter, 2009). In the end the person should be faster and/or the solution should be more accurate. Strategy use can be seen as an indicator for the state of development of a mathematical concept. Adaptive strategy use necessitates shifts between: (a) calculating problems in the general mode (b) investing some time and effort to search for shortcut options, and (c) using a shortcut option. We are interested in factors that can tip the balance on the exploitation-exploration continuum. Experts know when to search for a new shortcut strategy and when not, children have to learn how much time and effort they want to spend for searching. Teachers etc. cannot sustainably take over the regulation of this dilemma calculating in standard way or flexible change strategies—they can only help children to calibrate the balance between flexibility vs. stability (or exploration vs. exploitation).

We have to acknowledge that the effects of spontaneously using a shortcut were small in many cases of the current experiments and the variability across students was large. This is to be expected when taking into account the difference between competence and performance (i.e., principle knowledge and application). Larger estimates of both procedural and conceptual knowledge have been obtained when knowledge was probed more directly (Prather and Alibali, 2009). Direct probing, however, does convey to the students that and which shortcut options exist. It is therefore not suitable when trying to measure the extent to which knowledge about a mathematical principle is applied spontaneously (cf. Haider et al., 2014). In addition, Robinson and Dubé (2012) have suggested that personality characteristics bridge between knowledge and application. They argued that some children have more positive attitudes toward accepting strategies that are highly efficient but are novel to their current strategy repertoire of algorithmic approaches. In a similar vein, (Guerrero and Palomaa, 2012) highlighted that some children change their strategies during calculation while some do not. Furthermore, children change their strategies for different reasons. It is not always the goal to choose the most efficient strategy (Newton et al., 2010) suggested that flexibility involves the use of strategies, which are considered the most appropriate for a given problem. They also discussed what "appropriate" means. It could be the most efficient or the most understandable strategy in a given situation. Which strategy in general is used depends on the problem, the numbers presented and other contextual, developmental, or personal factors (Newton et al., 2010; Guerrero and Palomaa, 2012). An U-shaped relationship between knowledge/understanding and variety of strategy use suggests that novices as well as experts may use a large variety of strategies (Siegler and Jenkins, 1989; Dowker et al., 1996). Experts like mathematic students used large numbers of appropriate strategies (Dowker et al., 1996) whereas children (novices) may use a large variety of appropriate and inappropriate strategies, because they have not yet acquired a small set of well-learned strategies (Dowker et al., 1996). In contrast to this assumption Newton et al. (2010) argued that low achieving students might be particularly appreciative and excited about a focus on multiple strategies to compare the possible ways to solve the problem and maximize the accuracy. Although the idea is prominent that an educational approach for low achieving children should promote routine mastery of a single well-thought solution strategy for a given type of problems (e.g.,Woodward and Baxter, 1997; Baxter et al., 2001). Future work should explore how students at different ability levels profit from sequences of problems allowing for different shortcuts based on the same mathematical principle.

In order to optimize the chances to measure spontaneous (i.e., no cues and no instruction) recruitment of knowledge about the commutativity principle we chose a paper-and-pencil test in the classroom in Experiment 2. Our informal observations suggest that children taking part in an eye tracking study on mental arithmetic appreciate that the measurement is (not only) about whether they solve the problems correctly, but also on how they solve them. The paper-and-pencil method was closer to usual test situations in the classroom. Children focused on being fast and accurate rather than on the fact that someone might be trying to assess *how* they solved the problems. Verschaffel et al. (2009) highlighted the importance of ecological validity for studies on adaptive expertise. We suggest that trial-by-trial process measures (as in our eye tracking experiment) and ecologically valid but less sensitive methods (as in Experiment 2) should be combined to convey the full picture. For instance, eye tracking can help to figure out whether increased time demands after a change in shortcut option reflect prolonged solution times or, alternatively, a mixture of prolonged solution times plus time invested in search for alternative shortcut options. Potentially, learners at different levels of expertise might differ in both the efficiency in spotting shortcuts as well as in using them. For instance, third graders might have discovered the options for the addends-compare shortcut relatively quickly even without a fitting warm-up condition.

In line with the research on adaptive expertise Verschaffel et al. (2009) or Star and Rittle-Johnson (2008) defined flexibility in problem solving as knowledge of multiple strategies and their relative efficiency. In addition to weighing different strategies according to their efficiency, students need to weigh the potential costs and benefits of flexible strategy usage. There are time costs of switching between strategies, once a shortcut option has been discovered (Lemaire and Lecacheur, 2010). Luwel et al. (2009) found longer response times but no reduced accuracy and the size of these switching costs varied as a function of the associative strength between a strategy and a particular problem. More importantly, there is a dilemma between (a) investing time and attention in order to spot potential shortcut options that might or might not exist and (b) using processing strategies readily available (e.g., Jepma and Nieuwenhuis, 2011). Thus, process measures that provide evidence on when, how and to what extent students invest in spotting and applying shortcuts (Haider and Rose, 2007) are necessary in order to better understand the bases of the transfer effect observed in Experiment 2. To illustrate the search process, we additionally used eye tracking assessment in the Experiment 1. On the one hand this is a more specific method than paper pencil and on the other hand we could measure the shift of attention. The eye tracking results are in line with the view of (Robinson and LeFevre, 2012). For discovering new strategies, children need to shift their attention to the relevant part of the problem. The eye movement patterns were different for the different shortcut strategies and fit to the points of interests of the according strategies.

ACKNOWLEDGMENTS

This work was supported by Grant FR 1471/12-1 from the Deutsche Forschungsgemeinschaft (DFG) as well as by the Berlin Cluster of Excellence Image Knowledge Gestaltung (www.interdisciplinary-laboratory.hu-berlin.de). Some of the results were presented at the Conference of Experimental Psychologists 2013 in Vienna, Austria. We thank Christine Arndt, Kate Könnecke and Maria Wirth for help with data collection.

SUPPLEMENTARY MATERIAL

The Supplementary Material for this article can be found online at: http://www.frontiersin.org/journal/10.3389/fpsyg.2014.00556/abstract

REFERENCES

Anderson, J. R., and Schunn, C. D. (2000). "Implications of the ACT-R learning theory: no magic bullets," in *Advances in Instructional Psychology: Educational Design and Cognitive Science*, ed R. Glaser (Mahwah, NJ: Erlbaum), 1–33.

Baroody, A. J., and Gannon, K. E. (1984). The development of the commutativity principle and economical addition strategies. *Cogn. Instr.* 1, 321–339. doi: 10.1207/s1532690xci0103_3

Baroody, A. J., Ginsburg, H. P., and Waxman, B. (1983). Children's use of mathematical structure. *J. Res. Math. Educ.* 14, 156–168. doi: 10.2307/748379

Baxter, J. A., Woodward, J., and Olson, D. (2001). Effects of reform-based mathematics instruction on low achievers in five third-grade classrooms. *Elem. Sch. J.* 101, 529–547. doi: 10.1086/499686

Bilalić, M., and McLeod, P. (2014). Why good thoughts block better ones. *Sci. Am.* 310, 74–79. doi: 10.1038/scientificamerican0314-74

Bilalić, M., McLeod, P., and Gobet, F. (2008a). Inflexibility of experts—reality or myth? Quantifying the Einstellung effect in chess masters. *Cogn. Psychol.* 56, 73–102. doi: 10.1016/j.cogpsych.2007.02.001

Bilalić, M., McLeod, P., and Gobet, F. (2008b). Why good thoughts block better ones: the mechanism of the pernicious Einstellung (set) effect. *Cognition* 108, 652–661. doi: 10.1016/j.cognition.2008.05.005

Byrnes, J. P., and Wasik, B. A. (1991). Role of conceptual knowledge in mathematical procedural learning. *Dev. Psychol.* 27, 777–786. doi: 10.1037/0012-1649.27.5.777

Canobi, K. H. (2009). Concept–procedure interactions in children's addition and subtraction. *J. Exp. Child Psychol.* 102, 131–149. doi: 10.1016/j.jecp.2008.07.008

Canobi, K. H., Reeve, R. A., and Pattison, P. E. (2002). Young children's understanding of addition concepts. *Educ. Psychol.* 22, 513–532. doi: 10.1080/0144341022000023608

Canobi, K. H., Reeve, R. A., and Pattison, P. E. (2003). Patterns of knowledge in children's addition. *Dev. Psychol.* 39, 521–534. doi: 10.1037/0012-1649.39.3.521

Cowan, R., and Renton, M. (1996). Do they know what they are doing? Children's use of economical addition strategies and knowledge of commutativity. *Educ. Psychol.* 16, 407–420. doi: 10.1080/0144341960160405

Dowker, A. (2014). Young children's use of derived fact strategies for addition and subtraction. *Front. Hum. Neurosci.* 7:924. doi: 10.3389/fnhum.2013.00924

Dowker, A., Flood, A., Griffiths, H., Harriss, L., and Hook, L. (1996). Estimation strategies of four groups. *Math. Cogn.* 2, 113–135. doi: 10.1080/135467996387499

Dubé, A. K., and Robinson, K. M. (2010). The relationship between adults' conceptual understanding of inversion and associativity. *Can. J. Exp. Psychol.* 64, 60–66. doi: 10.1037/a0017756

ErEl, H., and Meiran, N. (2011). Mindset changes lead to drastic impairments in rule finding. *Cognition* 119, 149–165. doi: 10.1016/j.cognition.2011.01.002

Ericsson, K. A. (2008). Deliberate practice and acquisition of expert performance: a general overview. *Acad. Emerg. Med.* 15, 988–994. doi: 10.1111/j.1553-2712.2008.00227.x

Ericsson, K. A., and Lehmann, A. C. (1996). Expert and exceptional performance: evidence of maximal adaptation to task constraints. *Annu. Rev. Psychol.* 47, 273–305. doi: 10.1146/annurev.psych.47.1.273

Frensch, P. A., and Haider, H. (2008). "2.31—transfer and expertise: the search for identical elements," in *Learning and Memory: A Comprehensive Reference*, ed H. B. John (Oxford: Academic Press), 579–596. Available online at: http://www.sciencedirect.com/science/article/pii/B9780123705099001777

Gaschler, R., Vaterrodt, B., Frensch, P. A., Eichler, A., and Haider, H. (2013). Spontaneous usage of different shortcuts based on the commutativity principle. *PLoS ONE* 8:e74972. doi: 10.1371/journal.pone.0074972

Gentner, D., and Toupin, C. (1986). Systematicity and surface similarity in the development of analogy. *Cogn. Sci.* 10, 277–300. doi: 10.1016/S0364-0213(86)80019-2

Guerrero, S. M., and Palomaa, K. (2012). First-grade methods of single-digit addition with two or more addends. *J. Res. Child. Educ.* 26, 1–17. doi: 10.1080/02568543.2012.633841

Haider, H., Eichler, A., Hansen, S., Vaterrodt, B., Gaschler, R., and Frensch, P. (2014). How we use what we learn in Math: an integrative account of the development of commutativity. *Frontline Learn. Res.* 2, 1–21. doi: 10.14786/flr.v2i1.37

Haider, H., and Frensch, P. A. (1996). The Role of Information Reduction in Skill Acquisition. *Cogn. Psychol.* 30, 304–337. doi: 10.1006/cogp.1996.0009

Haider, H., and Rose, M. (2007). How to investigate insight: a proposal. *Methods* 42, 49–57. doi: 10.1016/j.ymeth.2006.12.004

Hannula, M. M., and Lehtinen, E. (2005). Spontaneous focusing on numerosity and mathematical skills of young children. *Learn. Instr.* 15, 237–256. doi: 10.1016/j.learninstruc.2005.04.005

Hannula, M. M., Lepola, J., and Lehtinen, E. (2010). Spontaneous focusing on numerosity as a domain-specific predictor of arithmetical skills. *J. Exp. Child Psychol.* 107, 394–406. doi: 10.1016/j.jecp.2010.06.004

Harel, A., Kravitz, D., and Baker, C. I. (2013). Beyond perceptual expertise: revisiting the neural substrates of expert object recognition. *Front. Hum. Neurosci.* 7:885. doi: 10.3389/fnhum.2013.00885

Hatano, G. (1988). Social and motivational bases for mathematical understanding. *New Dir. Child Adolesc. Dev.* 1988, 55–70. doi: 10.1002/cd.23219884105

Hatano, G., and Oura, Y. (2003). Commentary: reconceptualizing school learning using insight from expertise research. *Educ. Res.* 32, 26–29. doi: 10.3102/0013189X032008026

Hiebert, J., and Wearne, D. (1996). Instruction, understanding, and skill in multidigit addition and subtraction. *Cogn. Instr.* 14, 251–283. doi: 10.1207/s1532690xci1403_1

Jepma, M., and Nieuwenhuis, S. (2011). Pupil diameter predicts changes in the exploration–exploitation trade-off: evidence for the adaptive gain theory. *J. Cogn. Neurosci.* 23, 1587–1596. doi: 10.1162/jocn.2010.21548

Koedinger, K. R., and Anderson, J. R. (1990). Abstract planning and perceptual chunks: elements of expertise in geometry. *Cogn. Sci.* 14, 511–550. doi: 10.1207/s15516709cog1404_2

Lemaire, P., and Lecacheur, M. (2010). Strategy switch costs in arithmetic problem solving. *Mem. Cogn.* 38, 322–332. doi: 10.3758/MC.38.3.322

Leys, C., Klein, O., Bernard, P., and Licata, L. (2013). Detecting outliers: do not use standard deviation around the mean, use absolute deviation around the median. *J. Exp. Soc. Psychol.* 49, 764–766. doi: 10.1016/j.jesp.2013.03.013

Logan, G. D. (1988). Toward an instance theory of automatization. *Psychol. Rev.* 95, 492–527. doi: 10.1037/0033-295X.95.4.492

Luchins, A. S. (1942). "Mechanization in problem solving: the effect of einstellung," in *Psychological Monographs*, Vols. 1–6, Vol. 54, ed J. F. Dashiell (Illinois, IL: The American Psychological Association), 1–95.

Luwel, K., Schillemans, V., Onghena, P., and Verschaffel, L. (2009). Does switching between strategies within the same task involve a cost? *Br. J. Psychol.* 100, 753–771. doi: 10.1348/000712609X402801

McMullen, J. A., Hannula-Sormunen, M. M., and Lehtinen, E. (2011). "Young children's spontaneous focusing on quantitative aspects and their verbalizations of their quantitative reasoning," in *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3 (Ankara: PME), 217–224.

Newton, K. J., Star, J. R., and Lynch, K. (2010). Understanding the development of flexibility in struggling algebra students. *Math. Think. Learn.* 12, 282–305. doi: 10.1080/10986065.2010.482150

Palmeri, T. J., Wong, A. C.-N., and Gauthier, I. (2004). Computational approaches to the development of perceptual expertise. *Trends Cogn. Sci.* 8, 378–386. doi: 10.1016/j.tics.2004.06.001

Prather, R. W., and Alibali, M. W. (2009). The development of arithmetic principle knowledge: how do we know what learners know? *Dev. Rev.* 29, 221–248. doi: 10.1016/j.dr.2009.09.001

Resnick, L. B. (1992). "From protoquantities to operators: building mathematical competence on a foundation of everyday knowledge," in *Analysis of Arithmetic For Mathematics Teaching*, eds G. Leinhardt, R. Putnam, and R. A. Hattrup (Hillsdale, NJ: L. Erlbaum Associates), 1–92.

Rittle-Johnson, B., Siegler, R. S., and Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: an iterative process. *J. Educ. Psychol.* 93, 346–362. doi: 10.1037/0022-0663.93.2.346

Robinson, K., and LeFevre, J.-A. (2012). The inverse relation between multiplication and division: concepts, procedures, and a cognitive framework. *Educ. Stud. Math.* 79, 409–428. doi: 10.1007/s10649-011-9330-5

Robinson, K. M., and Dubé, A. K. (2009). Children's understanding of addition and subtraction concepts. *J. Exp. Child Psychol.* 103, 532–545. doi: 10.1016/j.jecp.2008.12.002

Robinson, K. M., and Dubé, A. K. (2012). Children's use of arithmetic shortcuts: the role of attitudes in strategy choice. *Child Dev. Res.* 2012, 10. doi: 10.1155/2012/459385

Rosenberg-Lee, M., Barth, M., and Menon, V. (2011). What difference does a year of schooling make? Maturation of brain response and connectivity between 2nd and 3rd grades during arithmetic problem solving. *Neuroimage* 57, 796–808. doi: 10.1016/j.neuroimage.2011.05.013

Selter, C. (2009). Creativity, flexibility, adaptivity, and strategy use in mathematics. *ZDM* 41, 619–625. doi: 10.1007/s11858-009-0203-7

Siegler, R. S. (1989). Hazards of mental chronometry: an example from children's subtraction. *J. Educ. Psychol.* 81, 497–506. doi: 10.1037/0022-0663.81.4.497

Siegler, R. S., and Jenkins, E. (1989). *How Children Discover New Strategies*, Vol. 14. Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Star, J. R., and Rittle-Johnson, B. (2008). Flexibility in problem solving: the case of equation solving. *Learn. Instr.* 18, 565–579. doi: 10.1016/j.learninstruc.2007.09.018

Verschaffel, L., De Corte, E., Gielen, I., and Struyf, E. (1994). "Clever rearrangement strategies in children's mental arithmetic: a confrontation of eye-movement data and verbal protocols," in *Research on Learning and Instruction of Mathematics in Kindergarten and Primary School*, ed J. E. H. Van Luit (Doetinchem: Graviant Publishing Company), 153–180.

Verschaffel, L., Luwel, K., Torbeyns, J., and Dooren, W. V. (2009). Conceptualizing, investigating, and enhancing adaptive expertise in elementary mathematics education. *Eur. J. Psychol. Educ.* 24, 335–359. doi: 10.1007/BF03174765

Waldmann, M. (2006). "Konzepte und Kategorien," in *Handbuch der Psychologie*, eds J. Funke and P. A. Frensch (Göttingen: Hogrefe Verlag), 283–293.

Wilkins, J. L., Baroody, A. J., and Tiilikainen, S. (2001). Kindergartners' understanding of additive commutativity within the context of word problems. *J. Exp. Child Psychol.* 79, 23–36. doi: 10.1006/jecp.2000.2580

Woodward, J., and Baxter, J. A. (1997). The effects of an innovative approach to mathematics on academically low-achieving students in inclusive settings. *Except. Child.* 63, 373–388.

Conflict of Interest Statement: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Received: 26 March 2014; paper pending published: 16 April 2014; accepted: 19 May 2014; published online: 10 June 2014.

Citation: Godau C, Haider H, Hansen S, Schubert T, Frensch PA and Gaschler R (2014) Spontaneously spotting and applying shortcuts in arithmetic—a primary school perspective on expertise. Front. Psychol. ***5****:556. doi: 10.3389/fpsyg.2014.00556*

This article was submitted to Cognition, a section of the journal Frontiers in Psychology.

Godau, C., Wirth, M., Hansen, S., Haider, H., & Gaschler, R. (2014). From Marbles to Numbers—Estimation Influences Looking Patterns on Arithmetic Problems. *Psychology*, *05*(02), 127–133. doi:10.4236/psych.2014.52020

Psychology
2014. Vol.5, No.2, 127-133
Published Online February 2014 in SciRes (http://www.scirp.org/journal/psych)

Scientific Research
http://dx.doi.org/10.4236/psych.2014.52020

From Marbles to Numbers—Estimation Influences Looking Patterns on Arithmetic Problems

Claudia Godau*, Maria Wirth, Sonja Hansen, Hilde Haider, Robert Gaschler
Department of Psychology, Humboldt Universität zu Berlin, Berlin, Germany
Email: *claudia.godau@hu-berlin.de

Received December 19th, 2013; revised January 18th, 2014; accepted February 16th, 2014

Flexibly spotting and applying shortcut options in arithmetic is often a major challenge for children as well as adults. Recent work has suggests that children benefit in terms of such flexibility from tasks requiring estimation or other operations with quantities that they cannot easily enumerate. Such tasks often require comparison of quantities by fixation and as such necessitate long-range eye movements, e.g. across the whole screen. We tested whether fixation patterns account for transfer from estimation to arithmetic tasks. Conceivably, participants who first solve estimation tasks are more flexible in spotting and applying shortcuts on later arithmetic tasks, because they stick to scanning the screen with long-range eye movements (which were necessary for solving the estimation task). To test this account, we manipulated the location of the marbles in an estimation task so that one group of participants had to make long-range eye movement, whereas another group did not need long-range eye movements to solve the task. Afterwards participants of both groups solved addition problems that contained a shortcut option based on the commutativity principle. We tested whether shortcut usage and fixation patterns in the arithmetic problems were influenced by the variant of the estimation task provided beforehand. The experiment allowed us to explore whether flexibility in spotting and using arithmetic shortcuts can be fostered by applying a prior task that induces flexible looking patterns. The results suggest that estimation tasks can indeed influence fixation patterns in a later arithmetic task. While shortcut search and application is reflected in fixation patterns, we did not obtain evidence for the reverse influence. Changed fixation patterns did not lead to higher shortcut usage. Thus, the results are in line with top-down accounts of strategy change: fixation patterns reflect rather than elicit strategy change.

Keywords: Mental Arithmetic; Estimation; Fixation Pattern; Transfer; Commutativity Principle

Introduction

For some elementary school children, arithmetic seems to be a tremendous challenge while for others it's a child's play. For all children it includes the discovery and use of general as well as abstract principles, which allow them to solve (apparently) hard problems easily and fast. Unfortunately the connection between understanding, identifying and applying shortcut options is at best moderate. Many researchers found large interindividual variability in the children's solution times and their use of shortcut strategies. For example, Dubé and Robinson (2010) found that 1/4 of the children did not use any shortcut at all. Robinson and Dubé (2012) later argued that children have different attitudes toward accepting strategies that are highly efficient but novel. For instance, from grade three onwards commutativity-based shortcuts are used spontaneously (Gaschler, Vaterrodt, Frensch, Eichler, & Haider, 2013). Children used a shortcut when subsequent addition problems contained the same addends in different order (e.g., 6 + 2 + 3 = ? followed by 2 + 6 + 3 = ?). Furthermore, spontaneous application of two different commutativity-based shortcuts correlated positively. However, while it could be documented that commutativity-based shortcuts were used spontaneously, the rate of usage was rather low. Apart from difficulties in spontaneously using shortcuts, past research has documented overgeneralization (e.g. Siegler & Stern, 1998). Once children started to use one shortcut, some tended to apply it irrespective of whether the underlying mathematical principle was applicable to the current arithmetic problem or not. This research showed that during elementary school the flexible use of shortcut strategies is not very balanced yet.

One potential way to foster adaptive flexibility in strategy usage (Verschaffel, Luwel, Torbeyns, & Dooren, 2009) might be to employ estimation tasks. By this, the mathematical principles and the corresponding shortcuts might be conveyed at early age. This might provide a head start for tackling the respective shortcuts in mental arithmetic. Several studies on commutativity have shown that children have at least some understanding of the concept of commutativity before entering school (Canobi, Reeve, & Pattison, 2003; Cowan & Renton, 1996; Resnick, 1992; Wilkins, Baroody, & Tiilikainen, 2001). Fur-

*Corresponding authors.

thermore, there is evidence for that toddlers develop an informal understanding of relations between objects in the real world before entering school (Baroody & Gannon, 1984; Baroody, Ginsburg, & Waxman, 1983; Canobi, Reeve, & Pattison, 2002; Gallistel & Gelman, 1992; Gilmore, McCarthy, & Spelke, 2010; Resnick, 1992; Siegler & Jenkins, 1989; Sophian, Harley, & Martin, 1995). This idea is based on Resnick's (1992) model of mathematical thinking, in which on the first level, the level of protoquantities, mathematical thinking is strictly object-bound.

Moreover, Sherman & Bisanz (2009) showed that working with non-symbolic material can encourage subsequent strategy use in symbolic equivalence problems. In this context, research hints at that estimation might positively influence exact calculation. Gilmore, McCarthy and Spelke (2007) found that even preschoolers were able to solve symbolic problems as long as they were instructed to just estimate the results, rather than to calculate the exact result. Along these lines, Hansen and colleagues (submitted) tested whether children would profit from an estimation task involving commutative arithmetic problems on later arithmetic problems. They confirmed the assumption that symbolic estimation increased the spontaneous spotting and applying of commutativity-based shortcuts in a later arithmetic task. Surprisingly, the positive effect seemed to be confined to actual usage of commutativity-based shortcuts (procedural knowledge). In a task measuring conceptual understanding of the commutativity principle, children showed liberalization in the response criterion rather than improved or maintained performance when before confronted with a commutativity-based estimation task.

Apparently, symbolic as well as non-symbolic estimation tasks support the use of commutativity-based shortcuts—but not by activating conceptual knowledge of the mathematical principle. This suggests to explore potential ways of transfer between the task formats that side-track conceptual knowledge. One potential account for the incoherent transfer results reported by Hansen et al. (submitted) and Sherman and Bisanz (2009) might be a transfer of eye movement patterns. Participants might profit from the estimation task in spotting and applying shortcuts in later arithmetic problems, because (a) long-range eye movements are helpful in both contexts and are (b) transferred from the estimation task to the arithmetic task. Specifically, short cut strategies that entail comparing addends across subsequent addition problems should necessitate unusually long eye movements. Potentially, such looking patterns —triggered by an estimation task—would still be present when later faced with an addition task and raise the chance that a child spots and applies the shortcut.

Our hypothesis was that a variant of an estimation task that requires long-range eye movements, would lead to a larger amount of shortcut usage in a later arithmetic task (as compared to an estimation task not requiring long-range eye movements). To test this hypothesis, we actively manipulated the eye movements in order to investigate the influence of long-range eye movements on the detection and application of possible shortcuts. Eye movement patterns can influence spatial reasoning by means of an implicit eye-movement-to-cognition link. For instance, Thomas and Lleras (2007) showed an implicit compatibility between spatial cognition and the eye movement. However, until now the influence of eye movement patterns (i.e., those triggered by an estimation task) on arithmetic problems presented later on has been neglected. Therefore, we contrasted (a) one experimental condition starting with an estimation task necessitating long-range eye movements (scattered group) with (b) another group of primary school children starting with an estimation task without such demands (centered group). As dependent measure we tested the extent to which children saved calculation effort by a commutativity-based shortcut in a mental arithmetic task presented afterwards. Comparing addends in subsequent arithmetic problems, one could avoid calculation on problems that presented the same addends as the predecessor problem (e.g., 6 + 2 + 3 = ? followed by 2 + 6 + 3 = ?), this shortcut we called *addends compare strategy*.

Method

Participants

Thirty-four elementary school children (16 females and 18 males) with an age range between 6 and 11 years took part in the experiment. The mean age of the children in the *scattered group* was 8.78 years (SD = .92), and in the *centered group* 8.47 years (SD = 1.02). The children attended second to fifth grade of various Berlin elementary schools, most of them in the fourth grade (50%). The children were randomly assigned to either the scattered group (18 children) or the centered group (16 children).

Materials

The experiment comprised two parts. In the first part, the children were presented with an estimation task that differed between the groups (scattered/centered). In the second part, both groups were presented with an arithmetic task. All material was computerized.

For the *estimation task* four sets of eight estimation problems were designed, depicting different quantities of marbles. The two sets for the centered group consisted of one quantity of marbles, which belonged to one fictional character (either "Tim" or "Lisa"). In order to provoke a small spatial range and low number of saccades in the *centered group*, the quantity of marbles was presented centrally. Children were asked to estimate if the character owned few or many marbles (see **Figure 1**). The two sets for the *scattered group* included two different quantities of marbles of which one belonged either to "Tim" or "Lisa". To trigger a larger spatial range and higher number of saccades (compared to the centered group), the quantities of marbles were presented at right and left edge of the presentation frame. The children had to indicate which of the characters owned more marbles or if both own the same amount of marbles.

The 12 *arithmetic problems* were presented in two groups of six simultaneously depicted problems on two consecutive screens (see **Table A1**). We presented six problems in black on grey background simultaneously on the screen. Digits were approximately .5 cm wide and 1 cm tall. The distance both between the lines and columns of digits was 5 cm. Each addition problem consisted of three addends between 2 and 9 (e.g., 6 + 2 + 3 = ?). Each number occurred only once in a problem. Small and large numbers were balanced across the different problems. Each screen contained two commutative problem-pairs, in which the addends compare strategy could be used—one problem and its repetition with a different order of the same addends. All other problems were filler problems, with no shortcut option.

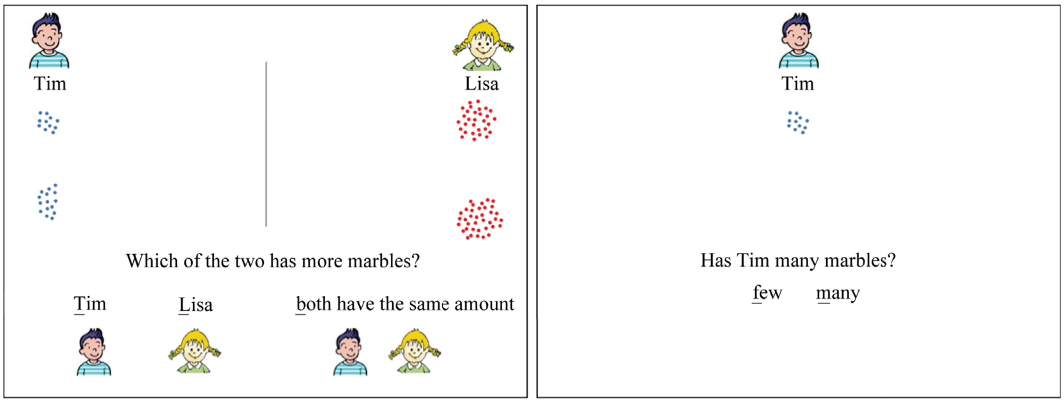


Figure 1.
Example of the estimation task for the scattered group (left) and the centered group (right).

Procedure

Each child was tested individually in the laboratory of the Department of General Psychology at Humboldt-University, Berlin. The children were seated in front of a 22 inch TFT computer monitor, which was equipped with the SMI RED stationary eye tracking system recording at 250 Hz. Children were asked to find a comfortable position (about 60 cm from the screen) and reminded to sit as still as possible. After five-point calibration, they received the instruction of the estimation task, which introduced them to fictional characters "Tim" and "Lisa" as owners of different quantities of marbles. They were then presented with an example problem. The children were reminded that the task did not require counting but only estimation and that they should answer as fast and as correct as possible.

In the estimation task, eight different estimation problems were presented consecutively. In a between subject design we wanted to manipulate the range and the number of the saccades during the estimation task to investigate the influence on discovering and using the commutativity shortcut strategy on later addition problems. For the centered group the marble quantities were presented centrally and the child had to decide if the pictured character had few or many marbles. For the scattered group the marble quantities were presented on the right and left edges of the screen and the child had to decide if one of the pictured characters had more marbles or if both had the same amount. After presenting each estimation problem for two seconds, the screen went blank. The time limit ensured that children had to rely on estimation. The experimenter entered the verbal answer of the child and started the next trial of the estimation task.

After completing the estimation task, the children calculated simple addition problems (arithmetic task). The children were instructed to work through the problems (six per screen) in strict order from top to bottom and not to leave out any. The experimenter entered the given answers so it was directly visible and remained visible when working on subsequent problems. After completing the first six problems, the experimenter started the second and last screen presenting the next six problems. Overall the procedure took about 10 minutes.

Results

The aim was to evaluate whether different eye movement patterns induced by different estimation tasks (scattered and centered) lead to different eye movement patterns and/or increased use of the addends-compare strategy in the arithmetic task.

Eye Movement Data

The eyetracking data of one child were not recorded correctly so we present the data of 33 children. In the analysis of the eye movements we focused on saccade distances, which were computed as Euclidean distance in angular degree. The saccade distances from commutative problems were compared to all other (non-commutative) addition problems. **Figure 2** depicts a bimodal distribution of saccade distances for the centered and scattered group. The bimodal character seemed more pronounced for commutative problems and especially so for the centered group.

For a more detailed analysis, we differentiated between horizontal and vertical saccades. To test if the two groups (scattered and centered group) differ in the saccade distances in the arithmetic task, we conducted ANOVAs for horizontal and vertical saccades separately. The results showed a significant difference between both groups for the horizontal saccade distances in the commutative problems, $F(1, 31) = 13.57$, $p = .001$. Surprisingly, the horizontal saccade distance for commutative problems was *lower* for the scattered as compared to the centered group (see **Table 1**). There were neither differences for the horizontal saccade distances for the non-commutative problems, $F(1, 31) = 1.33$, $p = .26$, nor between-group differences for the vertical saccade distances for non-commutative problems, $F(1, 31) = .65$, $p = .43$, and the commutative problems, $F(1, 31) = .780$, $p = .38$.

Additionally, we conducted a 2 (horizontal versus vertical saccades) × 2 (commutative versus non-commutative problems) × 2 (condition: scattered versus centered) ANOVA. As suggested by **Table 1**, the horizontal saccade distances were longer than the vertical saccade distances (main effect for horizontal versus vertical saccade distances, $F(1, 31) = 12605.84$, $p < .001$; $\eta p^2 = .998$). The ANOVA did not show a significant overall difference in the saccade distance for the commutative versus non-commutative problems, $F(1, 31) = 1.35$, $p = .25$; $\eta p^2 = .042$. However, we found a significant interaction of commutative versus non-commutative problems and condition, $F(1, 31) = 10.64$, $p = .01$; $\eta p^2 = .255$. This indicates that condition had an impact on the difference of saccade distances for commutative

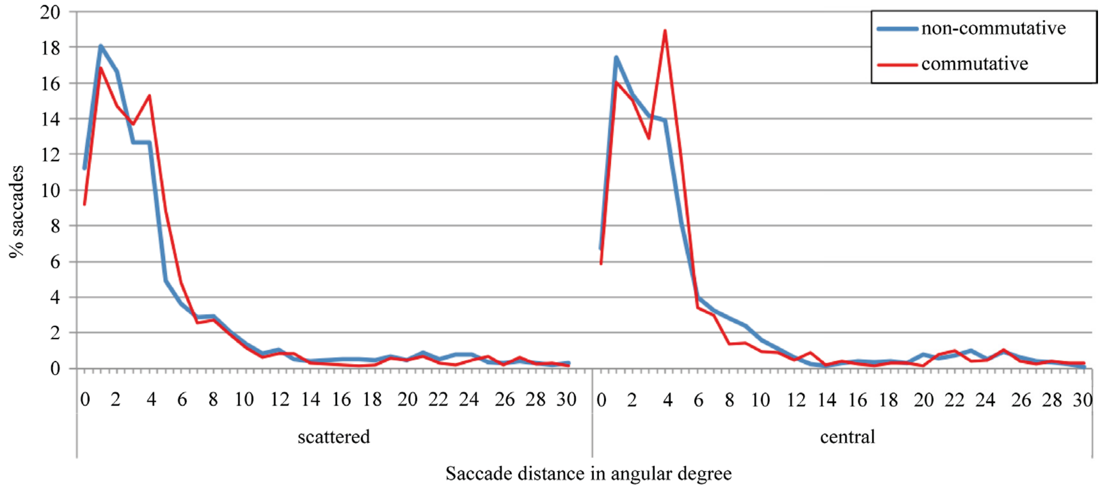


Figure 2.
Distribution in percent of saccade distances in angular degree for commutative and non-commutative problems for scattered group (left) and centered group (right).

Table 1.
Mean saccade distance horizontal and vertical for commutative and non-commutative problems analysed for condition.

Condition	Saccade distance horizontal		Saccade distance vertical	
	Non-commutative problems	Commutative problems	Non-commutative problems	Commutative problems
Scattered	4.38	4.11	2.28	2.24
Centered	4.24	4.41	2.32	2.3
Total	4.31	4.24	2.30	2.27

and non-commutative problems. Furthermore, this impact was different for horizontal and vertical saccade distances, because the 2-fold interaction was significant, too, $F(1, 31) = 15.28$, $p < .001$; $\eta p^2 = .330$. We did not obtain a main effect for condition, F < 1. In conclusion we manipulated the eye-movement in the estimation task (scattered and centered) and found different eyemovement patterns in the arithmetic task presented afterwards. For conclusions concerning the strategy use we present now the results of the solving times in the arithmetic task.

Solving Times

The solving time comprises the time between responding to one arithmetic problem and the time of the key press of the experimenter entering the verbalized answer to the subsequent problem (first key in entering the child's answer). **Figure 3** suggests that children in both experimental conditions benefitted from the addends-compare strategy. In line with exploiting the commutativity principle, children were faster when faced with the same addends in altered order for a second time on subsequent addition problems. The prior problem needed to be calculated conventionally, whereas the second problem of the commutativity pair consisted of the same addends in a different order. Supposed a child first calculated 6 + 3 + 2 and then 6 + 2 + 3 it would not need to calculate when confronted with the second problem—if it used the addends compare strategy. Across experimental conditions, a pairwise comparison of these two problems showed a significant benefit in solving times for commutative compared to their preceding non-commutative problems $t(32) = 3.37$, $p = .01$. Note that mean solving times of the filler problems were higher, 9.82 sec for the scattered group and 10.04 sec for the centered group. The comparison we focused on can thus be regarded as a conservative estimate of the usage of the addends compare strategy.

To test the difference between the scattered group and the centered group in solving times, we conducted a 2 (commutativity: commutative vs. non-commutative problems) × 2 (condition: scattered vs. centered group) ANOVA. As suggested by **Figure 3**, we obtained a significant main effect of commutativity, $F(1, 31) = 10.74$, $p = .01$, $\eta p^2 = .26$, but no effect of condition and no interaction effect of commutativity and condition (*Fs* < 1). The error rate for the twelve arithmetic problems was 10.2% and individual number of errors ranged from 0 to 6 (*mean* = 1.18; *SD* = 1.29). Note that children of different age seemed to profit to a similar extent from the addends compare strategy. We obtained no correlation between age and the benefit in solving times on commutative as compared to non-commutative problems.

Discussion

Prior work (Obersteiner et al., 2013; Hansen et al., submitted) indicates that estimation can positively influence subsequent calculation tasks. Our assumption was that this influence might in part be based on transfer of eye movement patterns. In an eyetracking experiment with primary school children, we contrasted a variant of an estimation task that did necessitate long-range eye movements with one that required central fixations. In particular, we were interested in how these two variants of an estimation task would influence later spontaneous usage of

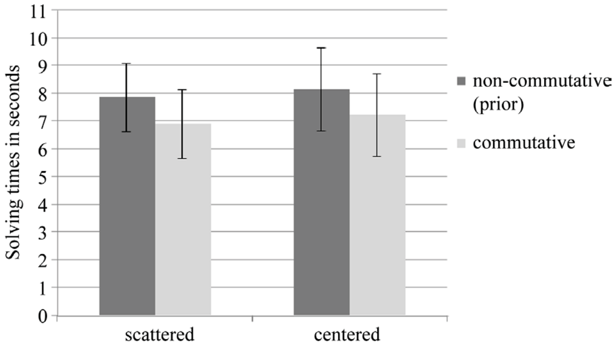


Figure 3.
The mean solving times in seconds per arithmetic problem for non-commutative (prior) problems (dark grey) in comparison to the commutative problems (light grey) for the scattered and the centered group. The error bar displays the 95% confidence interval of the comparison with each other.

commutativity-based shortcuts in mental arithmetic. While we did find an effect of estimation task variant on eye movement patterns in the arithmetic task, we did not obtain any effect on shortcut usage. Children of both experimental groups profited from the commutative addition problems to the same extent.

Surprisingly, the effect of condition on eye movements in the arithmetic task was opposite to our expectations. At present we can only state that we did see an effect of the estimation task on eye movements in a later calculation task and speculate about the reasons for its unexpected direction. The estimation tasks in the scattered and centered group were designed to elicit long-range eye movements to a different extent. To minimize these saccades in the centered group we presented only one character (Tim/Lisa) centrally and slightly changed the question to, whether Tim/Lisa has many or few marbles. In comparison, the scattered group was asked in the estimation task whether Tim or Lisa has more marbles. Both tasks required some form of estimation, but the children in the centered group had to compare the centrally depicted marbles with their own concept of few or many. To the scattered group, however, the comparison array was presented simultaneously on the other side of the computer screen. Presumably, in the centred group one additional step of processing was needed (i.e., "What do I consider as few ore many marbles?"). Accordingly, one limitation of the current research is that the two variants of the estimation task not only differed considerably in the type of saccades demanded, but also in other cognitive processes involved in generating an answer. The requirements in the centered condition might have supported flexibility in thinking in this group, because the criterion to decide between *few* and *many* can adaptively change between the problems (e.g. in one problem a child might consider 8 marbles as *many*, but after seeing 20 marbles, 8 seem to be only a *few*). Presumably, such demanding comparisons might lead to more comparisons between numbers on the screen once the addition problems are being presented. This in turn, might be the reason for the centred group executing longer saccades. Yet, these differences in fixation patterns did not lead to differences in spotting and applying shortcut options. This is coherent with models emphasizing the role of top-down decisions on strategy change in skill acquisition. For instance, Haider and Frensch (1999) suggested that changes in fixation patterns in a skill acquisition task involving a shift towards a more efficient strategy reflect the voluntary decision to change the strategy. Changes in fixation patterns might often reflect rather than cause changes in processing strategy. This is in line with the concept of adaptive expertise (Verschaffel et al., 2009), according to which learners need to autonomously regulate whether (a) to solve an arithmetic problem in a standard way or to (b) search for/apply a shortcut.

Past work (e.g. Gaschler et al., 2013; Godau et al., submitted) showed that flexible strategy use is mirrored in fixation patterns and that different commutativity-based shortcuts are mirrored in different fixation patterns. For the use of the addends-compare strategy, the eye movement pattern showed that the children looked back to the preceding problem featuring the same addends in different order. While using a shortcut strategy is reflected in the corresponding fixation patterns, influencing fixation patterns does not necessarily lead to changes in arithmetic strategies. Future studies might test more direct ways to trigger flexibility in calculation strategies by inducing flexibility in fixation patterns. Visual cues can help the learner attend to and notice relevant information in the problem, which they previously may have ignored (Madsen, Rouinfar, Larson, Loschky, & Rebello, 2013). In their study, Madsen and colleagues (2013) found that inducing participants to look at helpful areas in physics problems and to ignore distracting areas when no visual cues are present is possibly a first step to support them to reason correctly about the problem. They also found transfer effects in that students could successfully answer and reason about related but different problems without cues (Madsen et al., 2013). Guiding attention to areas with regard to contents is one point; we, on the contrary, wanted to focus more on the eye movement as such, without special attention on content. For further investigations concerning a direct intervention to support flexible strategy use, the eye movement pattern could also be manipulated in a calculation task. During solving the arithmetic problems, children who have more long distance saccades (e.g. by presenting distractors in changing corners of the screen) maybe recognize the shortcut strategy more often.

In summary, the current results suggest that estimation problems can indeed influence fixation patterns in a later arithmetic task. While shortcut search and application is reflected in fixation patterns, we did not obtain evidence for the reverse influence. Changed fixation patterns did not lead to higher shortcut usage. Thus, the results are in line with top-down accounts of strategy change: fixation patterns reflect rather than elicit strategy change (cf. Haider & Frensch, 1999).

Author Note

The first two authors shared first authorship and contributed equally to this work. Claudia Godau, Humboldt-Universität zu Berlin, Germany; Maria Wirth, Universität Leipzig, Germany; Sonja Hansen and Hilde Haider, Universität zu Köln, Germany; Robert Gaschler, Universität Koblenz-Landau, Germany. This work was supported by Grant FR 1471/12-1 from the Deutsche Forschungsgemeinschaft (DFG, www.dfg.de/) as well as by the Berlin Cluster of Excellence Image Knowledge Gestaltung (www.interdisciplinary-laboratory.hu-berlin.de).

REFERENCES

Baroody, A. J., & Gannon, K. E. (1984). The development of the commutativity principle and economical addition strategies. *Cognition and Instruction, 1,* 321-339.

http://dx.doi.org/10.1207/s1532690xci0103_3

Baroody, A. J., Ginsburg, H. P., & Waxman, B. (1983). Children's use of mathematical structure. *Journal for Research in Mathematics Education, 14,* 156-168. http://dx.doi.org/10.2307/748379

Canobi, K. H., Reeve, R. A., & Pattison, P. E. (2002). Young children's understanding of addition concepts. *Educational Psychology, 22,* 513-532. http://dx.doi.org/10.1080/01443410220000023608

Canobi, K. H., Reeve, R. A., & Pattison, P. E. (2003). Patterns of knowledge in children's addition. *Developmental Psychology, 39,* 521-534. http://dx.doi.org/10.1037/0012-1649.39.3.521

Cowan, R., & Renton, M. (1996). Do they know what they are doing? Children's use of economical addition strategies and knowledge of commutativity. *Educational Psychology, 16,* 407-420. http://dx.doi.org/10.1080/0144341960160405

Dubé, A. K., & Robinson, K. M. (2010). The relationship between adults' conceptual understanding of inversion and associativity. *Canadian Journal of Experimental Psychology/Revue Canadienne de Psychologie Expérimentale, 64,* 60-66. http://dx.doi.org/10.1037/a0017756

Gallistel, C. R., & Gelman, R. (1992). Preverbal and verbal counting and computation. *Cognition, 44,* 43-74. http://dx.doi.org/10.1016/0010-0277(92)90050-R

Gaschler, R., Vaterrodt, B., Frensch, P. A., Eichler, A., & Haider, H. (2013). Spontaneous usage of different shortcuts based on the commutativity principle. *PLoS ONE, 8,* Article ID: e74972. http://dx.doi.org/10.1371/journal.pone.0074972

Gilmore, C. K., McCarthy, S. E., & Spelke, E. S. (2007). Symbolic arithmetic knowledge without instruction. *Nature, 447,* 589-591. http://dx.doi.org/10.1038/nature05850

Gilmore, C. K., McCarthy, S. E., & Spelke, E. S. (2010). Non-symbolic arithmetic abilities and achievement in the first year of formal schooling in mathematics. *Cognition, 115,* 394-406. http://dx.doi.org/10.1016/j.cognition.2010.02.002

Godau, C., Haider, H., Hansen, S., Vaterrodt, B., Schubert, T., Frensch, P. A., & Gaschler, R. (2013). Increasing the usage of an arithmetic shortcut by offering an easy-to-find shortcut based on the same mathematical principle. Manuscript submitted for publication.

Haider, H., & Frensch, P. A. (1999). Eye movement during skill acquisition: More evidence for the information-reduction hypothesis. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 25,* 172-190. http://dx.doi.org/10.1037/0278-7393.25.1.172

Hansen, S., Haider, H., Eichler, A., Gaschler, R., Godau, C., & Frensch, P. A. (2013). Fostering formal commutativity knowledge with approximate arithmetic. Manuscript submitted for publication.

Madsen, A., Rouinfar, A., Larson, A. M., Loschky, L. C., & Rebello, N. S. (2013). Can short duration visual cues influence students' reasoning and eye movements in physics problems? *Physical Review Special Topics—Physics Education Research, 9,* Article ID: 020104. http://dx.doi.org/10.1103/PhysRevSTPER.9.020104

Obersteiner, A., Reiss, K., & Ufer, S. (2013). How training on exact or approximate mental representations of number can enhance first-grade students' basic number processing and arithmetic skills. *Learning and Instruction, 23,* 125-135. http://dx.doi.org/10.1016/j.learninstruc.2012.08.004

Resnick, L. B. (1992). From protoquantities to operators: Building mathematical competence on a foundation of everyday knowledge. In G. Leinhardt, R. Putnam, & R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching*. Hillsdale, NJ: L. Erlbaum Associates.

Robinson, K. M., & Dubé, A. K. (2012). Children's use of arithmetic shortcuts: The role of attitudes in strategy choice. *Child Development Research, 2012,* 10. http://dx.doi.org/10.1155/2012/459385

Sherman, J., & Bisanz, J. (2009). Equivalence in symbolic and non-symbolic contexts: Benefits of solving problems with manipulatives. *Journal of Educational Psychology, 101,* 88-100. http://dx.doi.org/10.1037/a0013156

Siegler, R. S., & Jenkins, E. (1989). *How children discover new strategies* (*Vol. xiv*). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Siegler, R. S., & Stern, E. (1998). Conscious and unconscious strategy discoveries: A microgenetic analysis. *Journal of Experimental Psychology: General, 127,* 377-397. http://dx.doi.org/10.1037/0096-3445.127.4.377

Sophian, C., Harley, H., & Martin, C. S. M. (1995). Relational and representational aspects of early number development. *Cognition and Instruction, 13,* 253-268. http://dx.doi.org/10.1207/s1532690xci1302_4

Thomas, L. E., & Lleras, A. (2007). Moving eyes and moving thought: On the spatial compatibility between eye movements and cognition. *Psychonomic Bulletin & Review, 14,* 663-668. http://dx.doi.org/10.3758/BF03196818

Verschaffel, L., Luwel, K., Torbeyns, J., & Dooren, W. V. (2009). Conceptualizing, investigating, and enhancing adaptive expertise in elementary mathematics education. *European Journal of Psychology of Education, 24,* 335-359. http://dx.doi.org/10.1007/BF03174765

Wilkins, J. L., Baroody, A. J., & Tiilikainen, S. (2001). Kindergartners' understanding of additive commutativity within the context of word problems. *Journal of Experimental Child Psychology, 79,* 23-36. http://dx.doi.org/10.1006/jecp.2000.2580

Appendix

Table A1.
The 12 arithmetic problems (commutative and non-commutative problems). The results printed in italics had to be calculated by the participants.

Screen 1		
3 + 5 + 4	=	12
4 + 9 + 8	=	21
4 + 8 + 9	=	21
6 + 2 + 5	=	13
9 + 7 + 2	=	18
2 + 9 + 7	=	18
Screen 2		
6 + 3 + 2	=	11
6 + 2 + 3	=	11
8 + 9 + 6	=	23
7 + 2 + 6	=	15
6 + 7 + 2	=	15
7 + 4 + 8	=	19

Godau, C., Haider, H., & Gaschler, R. (submitted). Commutativity at first glance - mathematical and perceptual principles in bar-graph processing. *Journal of Cognitive Psychology, April 2014*.

Commutativity at first glance - mathematical and perceptual principles in bar-graph processing

Claudia Godau[1/2], Hilde Haider[3], & Robert Gaschler[2/4]

1 Humboldt-Universität Berlin

2 Cluster of Excellence: Image Knowledge Gestaltung. An Interdisciplinary Laboratory, Berlin

3 Universität Köln

4 Universität Koblenz-Landau

Author Note

This work was supported by the Berlin Cluster of Excellence Image Knowledge Gestaltung (www.interdisciplinary-laboratory.hu-berlin.de).

Correspondence concerning this article should be addressed to Claudia Godau, Department of Psychology, Humboldt Universität zu Berlin, Unter den Linden 6, D-10099 Berlin, Germany. E-mail: claudia.godau@hu-berlin.de.

We thank Sabrina Finger, Nadine Kaiser and Susanne Splitt for help with data collection.

Abstract

Bar graphs represent numerical quantities in an analog format. Their design can involve arbitrary grouping decisions (i.e., which column next to which). Research with preschool children suggests that knowledge about some mathematical principles relevant in arithmetic has precursors estimation. This poses the question whether processing of graphical representations is influenced by knowledge about such principles. With a sample of young adults we explore whether commutativity effects are present in the processing of briefly presented bar graphs. Similar to effects with numerical material, our results demonstrate a commutativity benefit for bar graphs that can be differentiated from perceptual effects such as mirror symmetry and pattern repetition. Overall, the results indicated that there might be a graphical equivalent of commutativity-based shortcuts in arithmetic.

Keywords: numerical cognition, estimation, commutativity, graph perception

Introduction

Data graphs have an important role in conveying results of scientific research and tend to replace tables especially in the natural sciences (Smith, Best, Stubbs, Johnston, & Archibald, 2000; Smith, Best, Stubbs, Archibald, & Roberson-Nay, 2002). This prompts the question how to predict the influence that graph design has on the representation of the results (Fischer, Dewulf, & Hill, 2005; Huestegge & Philipp, 2011). On the one hand, research on the estimation of quantities from bar graphs has suggested that perceptual properties are the most important determinants of graph processing (Meyer, Taieb, & Flascher, 1997). For instance, principles of perception such as grouping by proximity and similarity (Kubovy & van den Berg, 2008) as well as the privileged processing of horizontal mirror symmetry (Boldt, Stürmer, Gaschler, Schacht, & Sommer, 2013; Dehaene et al., 2010; Duñabeitia, Molinaro, & Carreiras, 2011) might determine what is perceived as elements of a graph and which numerical estimations result. On the other hand, graphs can be attributed a particularly short link to number representations, especially to analog representations of numerosity (Gallistel & Gelman, 1992, 2000). While relational properties of numbers in tables have to be computed – often demanding time and effort – such properties can directly be read-off from graphs (Schnotz & Bannert, 2003). On the other hand a wealth of research has confirmed that already toddlers develop an informal understanding of relations between objects in the real world before entering school (e.g. Baroody & Gannon, 1984; Gilmore, McCarthy, & Spelke, 2010; Resnick, 1992). Children can use the representation of approximate number to perform addition and subtraction even prior learning arithmetic (Gilmore, McCarthy, & Spelke, 2007). The commutativity principle is one relational property that students seem to be using spontaneously from primary school onwards (Gaschler, Vaterrodt, Frensch, Eichler, & Haider, 2013). When subsequent addition problems consist of the same addends in changed order (e.g. 5 +8 + 6 = ? after 8 + 6 + 5 = ?), students in many cases spontaneously notice and exploit the complete overlap in addends. The similarity amongst the addition problems might be noticed on the level of numbers – students realize that the same numbers are involved in both problems. Additionally, however, it is conceivable that overlap is realized on the level of graphical units. As the font and format are not changed, identical numbers are paralleled by identical graphical units. It is conceivable that students might not only spontaneously spot and apply shortcuts based on mathematical principles when confronted with arithmetic problems presented in numbers, but also on more general principles involving the processing of graphical elements. Therefore, it is relevant to ask which principles of the spontaneous usage of shortcuts relevant in arithmetic problems presented in a numerical format also apply for problems presented in data graphs such as bar graphs.

Here we explore for the case of bar graphs whether there might be a graphical equivalent of commutativity-based shortcuts that are spontaneously applied when addition problems are presented in numbers. Our task was similar to the task that voters in countries with proportional representation are faced with at the evening of an election day. Hour after hour, estimates on the votes being counted so far are presented in the media in bar graphs and similar graphs. As different coalitions of parties might result, it is key to check the balance between either of the two sums of votes that the parties in the two opposing camps could contribute to a potential coalition. Does the sum of the number represented in the bar graphs on the left outnumber the sum of the numbers in the bar graphs on the right?

Methods

Participants

In the experiment 24 (18 female, 5 male, 1 not specified) undergraduate university students of the Humboldt Universität zu Berlin participated and received partial course credit in exchange for their participation. 87.5% were right handed. The age ranged from 18 to 49 and the mean age was 25.1 years.

Materials

On each trial participants were briefly presented a bar graph (see Figure 1). The task was to indicate as quickly and accurately as possible with a key press, whether or not the sum of the three bars presented on the left of the graph (left triplet) equals the sum of the bars presented on the right (right triplet). A series of 108 different combinations of diagrams were employed in the study, 36 for each of the three conditions: (1) unequal sum, (2) equal sums with different addends and (3) same sum for the same addends.

(1) For the equal addends condition, we produced diagrams on the basis of the numbers 3, 5, 8. This allowed 6 different number combinations, which then led to 36 combinations of the conducted diagrams, when they are presented next to each. For the unequal addends and equal sums conditions diagrams on the basis of the numbers 3, 5 and 8

were used for the left side of the diagram. (2) For the right group of columns of the diagrams in the equal sum condition, six different number combinations were used which added up to 16 as well: (5, 6, 5); (7, 3, 6); (10, 5, 1); (2, 6, 8); (4, 7, 5); (6, 1, 9). (3) For creating the right triplet in diagrams of the unequal sums condition, six different sums were used: (2 + 7 + 4 = 13); (7 + 4 + 3 = 14); (4 +3 + 8 = 15); (6 + 8 + 3 = 16); (4 + 9 + 5 = 18); and (5 + 9 + 5 = 19). We took care to homogenize the standard deviation of the addends across problems (M = 2.51), and avoided cases where two of the three bars in a triplet were identical or highly similar.

Unequal sum　　　　Equal sum　　　　Equal addends

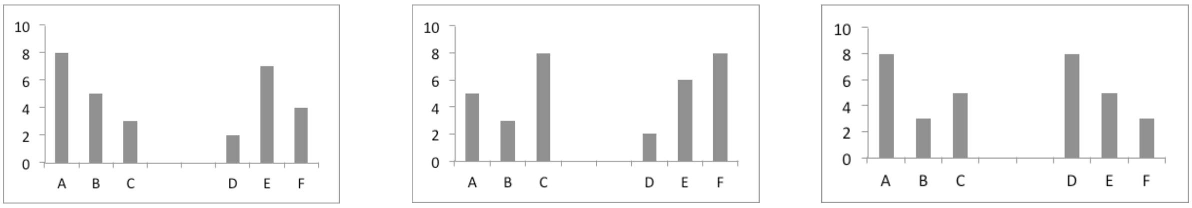


Figure 1. Example of the task for the condition unequal sum (left), the equal sum (center) and equal addends (right).

Procedure

The participants were seated at approximately 60 cm distance of a 19 inch TFT screen in a room equipped for group testing (computers visually separated by boards to avoid distraction). Diagrams were presented for two seconds maximum and erased as soon as a response was registered. If a participant had not responded within the maximum presentation time of two seconds on a trial, the diagram was erased and a five seconds interval with a blank screen was provided to complete the response. The experiment took about 20 minutes per participant.

While the three main conditions were *unequal sums* (1), *equal sums* with different addends (2) and equal sums with *equal addends* (3), we differentiated the trials further in the condition equal sums with equal addends for follow-up analyses: (4) some bar graphs consisted of two triplets of identical bars in the exact *same order.* Others (5) consisted of triplets with horizontally *mirrored* order (cf. Boldt et al., 2013). The same order and the mirror order diagrams were excluded from the main analyses and explored in the follow-up analysis. Of the 108 diagrams, 36 each belonged to the unequal and the equal sums condition. The Equal addends condition consisted of 24 diagrams, the special cased same order and mirrored order were instantiated by six diagrams each.

Results

Figure 2, shows the percent of correct responses per conditions as bars together with the mean reaction time (as points). In line with a commutativity benefit, the percent correct responses is highest and the RT is lowest in the equal addends trials. Percent of correct responses led to significant main effect of condition in the within subjects ANOVA, $F(2, 46) = 53.89$, $p < .001$, $\eta_p^2 = .70$. Percent correct responses were significantly higher in the equal addends as compared to the equal sum condition, $t(23) = -12.04$, $p < .001$. The equal sums condition was significantly worse than the unequal sums condition, $t(23) = -4.44$, $p < .001$. In fact, participants were below chance when judging diagrams consisting of two triplets of different bars that sum up to the same quantities.

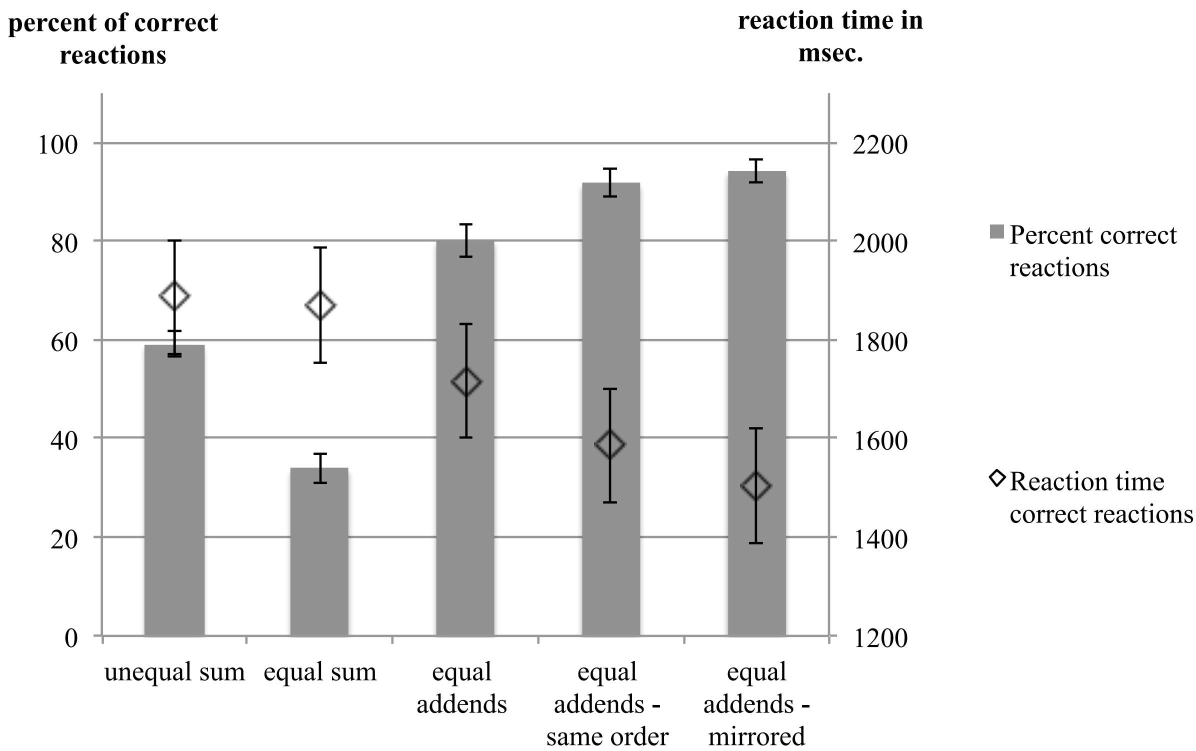


Figure 2. The percent of correct reactions (bars in dark grey) for all five conditions. The error bar displays the standard error of the mean. The points (rhombs) show the mean reaction time in milliseconds of the correct reaction for each condition, presented on the right axis of ordinates. The error bar displays the confidence interval based on Loftus and Masson (1994).

The ANOVA on RT revealed a significant effect of condition as well, $F(2, 46) = 5.52$, $p = .01$, $\eta_p^2 = .19$. Equal addends trials led to shorter responses than equal sums trials, $t(23) = -2.8$, $p = .01$, and also differed from the unequal sums trials, $t(23) = -3.1$, $p = .01$. Reaction times did not differ significantly between the unequal sum and the equal sum trials, $t(23) = -.27$, $p = .79$. Note that this was an aspect that changed when only the RT of correct trials were taken into account, $t(23) = 5.3$, $p = .001$ (compare Figure A1).

In a follow-up analysis we now also included the special cases of the equal addends condition: diagrams with identical columns in identical order and mirror-reversed triplets. The ANOVA with all five conditions showed a significant effect of condition for percent correct responses, $F(4, 92) = 88.83$, $p < .001$, $\eta_p^2 = .79$, as well as for RT, $F(4, 92) = 8.65$, $p < .001$, $\eta_p^2 = .27$.

Finally we contrasted the three conditions that all feature diagrams with two triplets of identical columns but differ with respect to how they are sorted (i.e. identical order, mirrored order, mixed order). Conditions only differed in perceptual characteristics (based on ordering of the columns) while the numbers displayed in the triplets were identical. We obtained a significant main effect of conditions for the percent correct responses, $F(2,46) = 11.64$, $p < .001$, $\eta_p^2 = .34$, as well as for the RTs, $F(2,46) = 4.36$, $p < .05$, $\eta_p^2 = .16$. On trials with same order and mirror order diagrams participants improved performance beyond the level obtained in trials with shuffled orders. Thus, perceptual grouping seems to play a role in addition to processes of bar graph perception that are tied to the represented quantities. Note that differentiating between trials with correct vs. incorrect response in the RT analysis did not change the pattern of results reported above (see Appendix).

Discussion

Often we cannot afford more than a glance to extract the gist of a bar graph. The present research demonstrates large influences of perceptual factors. Potentially arbitrary design decisions such as the exact sorting of columns can have a large impact on the numerical information retrieved from the graph. In addition, the results suggest that knowledge about mathematical principles, such as commutativity, does not only spontaneously manifest when people work on arithmetic problems with numerical material (e.g. Gaschler et al., 2013; Haider et al., 2014), but also when quantities are presented in an analog format in bar graphs.

We found a supportive effect of commutativity during graph processing. It is challenging to compare the sum of three bars on the right side to the left side, if the graph is only presented for 2 sec. In fact, also intuitive judgments in a complex domain are based on the perception of geometric features of the relevant information (Meyer et al., 1997). The percentage of correct responses of the control condition equal sum was high, but for the condition equal addends, in which a commutativity shortcut could be used the correct reactions and the reaction times decreased significantly. It is, first of all, likely that this phenomenon implies that commutativity is a perceptual property, which are the most important determinants of graph processing (Meyer et al., 1997).

Prior knowledge as well as graph design has an impact on the information processing regarding the presentation of scientific results in graphs. The supportive effect of the well-known arithmetic principle of commutativity is shown in less correct reactions and faster reactions times. Additionally, we found a larger effect of commutativity in the subconditions, in which the Gestalt principles were used. This could be interpreted as a supportive form of visualization used in the graphs, which supports the construction of a task-appropriate mental model (Schnotz & Bannert, 2003). One potentially limitation pertains to the fact that the results concerning the Gestalt principles are difficult to interpret, because the number of trials was rather low. This account is problematic, however the results are still significant and the effect size was moderate, this could support a strong effect.

Fischer and colleagues (2005) showed that designing optimal graphs can benefit from research into number representations. Therefore, he provided evidence that three effects concerning numerous representation are also supporting graph comprehension. We additionally showed that also an arithmetic principle like commutativity influenced the processing of graphs. In the graphs, no additionally numbers were presented such that the preverbal system of counting and arithmetic reasoning may have been addressed. In this system, numerosities are represented by magnitudes which are generated rapidly but inaccurately (Gallistel & Gelman, 1992, 2000). Our results demonstrate a commutativity benefit for bar graphs also for accuracy.

In summary, the reported findings suggest that there might be a graphical equivalent of commutativity-based shortcuts that are spontaneously applied when addition problems are presented in numbers. Future work should focus on children and whether the graphical equivalent of commutativity-based shortcuts will be used during the learning process. Haider and colleagous (2014) described that procedural and conceptual knowledge need to be integrated to a concept of commutativity and in which grade this process is going to start. Although some research exists that supports that graphs could hinder or support constructing mental model during the learning process, depending on (in-) appropriate design (Schnotz & Bannert, 2003).

References

Baroody, A. J., & Gannon, K. E. (1984). The development of the commutativity principle and economical addition strategies. *Cognition and Instruction*, *1*(3), 321–339. doi:10.1207/s1532690xci0103_3

Boldt, A., Stürmer, B., Gaschler, R., Schacht, A., & Sommer, W. (2013). Get out of here, quick! Problems with transparent labels on glass doors. *Journal of Experimental Psychology: Applied*, *19*(3), 241–253. doi:10.1037/a0034079

Dehaene, S., Nakamura, K., Jobert, A., Kuroki, C., Ogawa, S., & Cohen, L. (2010). Why do children make mirror errors in reading? Neural correlates of mirror invariance in the visual word form area. *NeuroImage*, *49*(2), 1837–1848. doi:10.1016/j.neuroimage.2009.09.024

Duñabeitia, J. A., Molinaro, N., & Carreiras, M. (2011). Through the looking-glass: Mirror reading. *NeuroImage*, *54*(4), 3004–3009. doi:10.1016/j.neuroimage.2010.10.079

Fischer, M. H., Dewulf, N., & Hill, R. L. (2005). Designing bar graphs: orientation matters. *Applied Cognitive Psychology*, *19*(7), 953–962. doi:10.1002/acp.1105

Gallistel, C. R., & Gelman, R. (1992). Preverbal and verbal counting and computation. *Cognition*, *44*(1–2), 43–74. doi:10.1016/0010-0277(92)90050-R

Gallistel, C. R., & Gelman, R. (2000). Non-verbal numerical cognition: from reals to integers. *Trends in Cognitive Sciences*, *4*(2), 59–65. doi:10.1016/S1364-6613(99)01424-2

Gaschler, R., Vaterrodt, B., Frensch, P. A., Eichler, A., & Haider, H. (2013). Spontaneous usage of different shortcuts based on the commutativity principle. *PLoS ONE*, *8*(9), e74972. doi:10.1371/journal.pone.0074972

Gilmore, C. K., McCarthy, S. E., & Spelke, E. S. (2007). Symbolic arithmetic knowledge without instruction. *Nature*, *447*(7144), 589–591.

Gilmore, C. K., McCarthy, S. E., & Spelke, E. S. (2010). Non-symbolic arithmetic abilities and achievement in the first year of formal schooling in mathematics. *Cognition*, *115*(3), 394–406. doi:10.1016/j.cognition.2010.02.002

Haider, H., Eichler, A., Hansen, S., Vaterrodt, B., Gaschler, R., & Frensch, P. (2014). How we use what we learn in Math: An integrative account of the development of commutativity. *Frontline Learning Research*, *2*(1). doi:10.14786/flr.v2i1.37

Huestegge, L., & Philipp, A. M. (2011). Effects of spatial compatibility on integration processes in graph comprehension. *Attention, Perception, & Psychophysics*, *73*(6), 1903–1915. doi:10.3758/s13414-011-0155-1

Kubovy, M., & van den Berg, M. (2008). The whole is equal to the sum of its parts: A probabilistic model of grouping by proximity and similarity in regular patterns. *Psychological Review*, *115*(1), 131 – 154.

Loftus, G. R., & Masson, M. E. J. (1994). Using confidence intervals in within-subject designs. *Psychonomic Bulletin & Review*, *1*(4), 476–490. doi:10.3758/BF03210951

Meyer, J., Taieb, M., & Flascher, I. (1997). Correlation estimates as perceptual judgments. *Journal of Experimental Psychology: Applied*, *3*(1), 3–20. doi:10.1037/1076-898X.3.1.3

Resnick, L. B. (1992). From protoquantities to operators: Building mathematical competence on a foundation of everyday knowledge. In G. Leinhardt, R. Putnam, & R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching*. Hillsdale, N.J.: L. Erlbaum Associates.

Schnotz, W., & Bannert, M. (2003). Construction and interference in learning from multiple representation. *Learning and Instruction*, *13*(2), 141–156. doi:10.1016/S0959-4752(02)00017-8

Smith, L. D., Best, L. A., Stubbs, D. A., Archibald, A. B., & Roberson-Nay, R. (2002). Constructing knowledge. The role of graphs and tables in hard and soft psychology. *Am Psychol*, *57*, 749–61.

Smith, L. D., Best, L. A., Stubbs, D. A., Johnston, J., & Archibald, A. B. (2000). Scientific graphs and the hierarchy of the sciences: A latourian survey of inscription practices. *Social Studies of Science*, *30*, 73–94. doi:10.1177/030631200030001003

Appendix

Analyses comparing RT for correct vs. incorrect responses only showed a tendency of an effect for the equal addends condition, $t(23) = -2.2$, $p = .05$. We are cautious to interpret it as the cell sizes differed substantially.

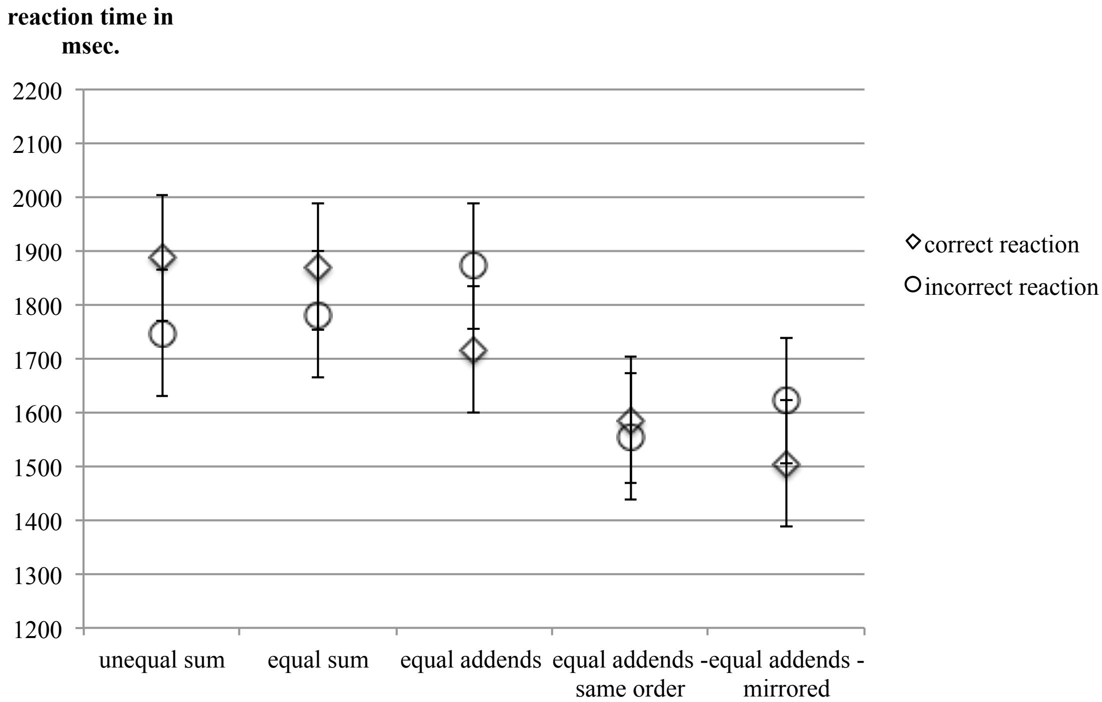


Figure A1. The mean reaction time in milliseconds of the correct reactions (rhomb) and for the incorrect reactions (points) for each condition are presented. The error bar displays the confidence interval based on Loftus and Masson (1994).